Millersville University Name Answer Key
Mathematics Department
MATH 365, *Ordinary Differential Equations*, **Test 3**
April 14, 2004

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. The Hermite ordinary differential equation is frequently encountered in quantum mechanics.

$$y'' - 2xy' + 4y = 0$$

(a) (8 points) Assuming there is a power series solution to the Hermite equation, find the recurrence relation which generates the coefficients in the power series.

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=1}^{\infty} a_n n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n+2=2}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (2n-4)a_n \right] x^n = 0$$

$$a_{n+2} = \frac{2n-4}{(n+2)(n+1)} a_n \quad \text{for } n = 0, 1, 2, \ldots$$

(b) (6 points) Find at least the first four nonzero terms (unless the series terminates sooner) of two linearly independent series solutions to the Hermite equation.

Let a_0 be arbitrary, then

$$a_2 = \frac{-4}{2\cdot 1} a_0 = -2a_0$$

$$a_4 = \frac{0}{4\cdot 3} a_2 = 0$$

$$a_6 = a_8 = \dots = a_{2n} = \dots = 0$$

Thus $y_1(x) = a_0(1 - 2x^2)$.

Now let a_1 be arbitrary, then

$$a_3 = \frac{-2}{3\cdot 2} a_1 = \frac{-1}{3} a_1$$

$$a_5 = \frac{2}{5\cdot 4} a_3 = \frac{-2}{5\cdot 4\cdot 3} a_1$$

$$a_7 = \frac{6}{7\cdot 6} a_5 = \frac{-2}{7\cdot 5\cdot 4\cdot 3} a_1$$

Thus $y_2(x) = a_1\left(x - \frac{x^3}{3} - \frac{2}{5\cdot 4\cdot 3}x^5 - \frac{2}{7\cdot 5\cdot 4\cdot 3}x^7 - \dots\right)$

2. (12 points) For the following ordinary differential equation, find all the singular points and determine which ones are regular singular points and which are irregular singular points.

$$(1-x)x^3y'' - (\sin x)y' + (1+x^2)y = 0$$

Let $P(x) = (1-x)x^3$, then the singular points of the ODE are $x_0 = 0$ and $x_0 = 1$.

<u>Case</u>: $x_0 = 0$

$$\lim_{x\to 0} x \frac{-\sin x}{(1-x)x^3} = -\lim_{x\to 0} \frac{\sin x}{(1-x)x^2}$$ which does not exist.

Thus $x_0 = 0$ is an irregular singular point.

<u>Case</u>: $x_0 = 1$

$$\lim_{x\to 1} (x-1) \frac{-\sin x}{(1-x)x^3} = \lim_{x\to 1} \frac{\sin x}{x^3} = \sin 1 < \infty$$

$$\lim_{x\to 1} (x-1)^2 \frac{1+x^2}{(1-x)x^3} = \lim_{x\to 1} \frac{-(1+x^2)(x-1)}{x^3} = 0 < \infty$$

Thus $x_0 = 1$ is a regular singular point.

3. (5 points each) Consider the following ordinary differential equation.

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0$$

For each of the following ordinary points determine the minimum radius of convergence for a power series solution centered at the ordinary point.

(a) $x_0 = 0$

Let $P(x) = x^2 - 2x - 3 = (x-3)(x+1)$. Thus the singular points are $x = -1$ and $x = 3$.

The radius of convergence $R \geq \min\{|-1-0|, |3-0|\} = 1$.

(b) $x_0 = 1$

The radius of convergence $R \geq \min\{|1-(-1)|, |3-1|\} = 2$.

(c) $x_0 = 2$

The radius of convergence $R \geq \min\{|2-(-1)|, |3-2|\} = 1$.

4. (13 points) Use Taylor's Method (repeated differentiation) to find the first four nonzero terms of the power series solution centered at $x_0 = 0$ to the initial value problem

$$\begin{aligned} y'' + xy' + (\sin x)y &= 0 \\ y(0) &= 1 = a_0 \\ y'(0) &= 0 = a_1 \end{aligned}$$

$$y''(0) = 2!\,a_2 = 0 \quad \Rightarrow \quad a_2 = 0$$

$$y^{(3)} = -y' - xy'' - (\cos x)y - (\sin x)y'$$

$$3!\,a_3 = -1 \quad \Rightarrow \quad a_3 = -1/3!$$

$$y^{(4)} = -y'' - xy''' - y'' + (\sin x)y - (\cos x)y' - (\cos x)y' - (\sin x)y''$$

$$= -2y'' - xy^{(3)} + (\sin x)y - (\sin x)y'' - 2(\cos x)y'$$

$$4!\,a_4 = 0 \quad \Rightarrow \quad a_4 = 0$$

$$y^{(5)} = -3y^{(3)} - xy^{(4)} + (\cos x)y + (\sin x)y' - (\sin x)y^{(3)} - (\cos x)y'' + 2(\sin x)y' - 2(\cos x)y''$$

$$5!\,a_5 = 3 + 1 = 4 \quad \Rightarrow \quad a_5 = \frac{1}{30}$$

Hence $y(x) = 1 - \dfrac{x^3}{6} + \dfrac{x^5}{30} + \dfrac{x^6}{180} + \cdots$

$$y^{(6)} = -(\sin x)y + 4(\cos x)y' + 6(\sin x)y'' - 4(\cos x)y^{(3)} - 4y^{(4)} - (\sin x)y^{(4)} - xy^{(}$$

$$6!\,a_6 = -4(-1) = 4 \quad \Rightarrow \quad a_6 = \frac{1}{180}$$

5. (10 points each) Find the general solution of the following Euler equations.

(a) $x^2y'' + 3xy' + 5y = 0$

Transformed equation: $\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + 5y = 0$

Characteristic equation: $r^2 + 2r + 5 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2}$$

$$r = -1 \pm 2i$$

$$y(x) = x^{-1}\left(c_1 \cos 2\ln x + c_2 \sin 2\ln x\right)$$

(b) $x^2y'' - 3xy' + 4y = 0$

Transformed equation: $\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 0$

Characteristic equation: $r^2 - 4r + 4 = 0$

$$(r-2)^2 = 0$$

$$r_1 = r_2 = 2$$

$$y(x) = (c_1 + c_2 \ln x)x^2$$

(c) $x^2y'' - 5xy' + 9y = 0$

Transformed equation: $\frac{d^2y}{dz^2} - 6\frac{dy}{dz} + 9y = 0$

Characteristic equation: $r^2 - 6r + 9 = 0$

$$(r-3)^2 = 0$$

$$r_1 = r_2 = 3$$

$$y(x) = (c_1 + c_2 \ln x)x^3$$

6. (8 points each) Consider the ordinary differential equation,

$$xy'' + 2xy' + 6e^x y = 0.$$

(a) Find the indicial equation associated with the regular singular point $x_0 = 0$.

$$\lim_{x \to 0} x \frac{2x}{x} = 0 = p_0$$

$$\lim_{x \to 0} x^2 \frac{6e^x}{x} = 0 = q_0$$

Indicial equation: $F(r) = r(r-1) + p_0 r + q_0 = 0$

$$r(r-1) = 0$$

(b) Find the exponents of singularity.

Exponents of singularity are the roots of the indicial equation.

$r_1 = 0$

$r_2 = 1$