Millersville University **Name** ______________________________

Department of Mathematics

MATH 365, *Ordinary Differential Equations*, Final Examination

April 28, 2009, 10:15AM–12:15PM

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (9 points) Solve the following initial value problem.

$$\begin{aligned} y' + 2t\,y &= 2t \\ y(0) &= 1 \end{aligned}$$

2. (9 points) A hard-boiled egg is removed from a pot of hot water. Initially, the egg's temperature is 180°F. After 15 minutes its temperature is 140°F. If the environment's temperature is 65°F, when will the egg have a temperature of 100°F?

3. (8 points) The value $x_0 = 0$ is an ordinary point for the ordinary differential equation below. Find the recurrence relation for a series solution to the ODE centered at $x_0 = 0$.

$$(1 + x^2)y'' + y = 0$$

4. (9 points) Solve the following initial value problem.

$$\begin{aligned} y'' - y' - 2y &= 2\sin 2t \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

5. (8 points) Suppose a population of birds obeys the logistic equation with threshold given below.

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right)P$$

where $k > 0$ and $0 < M < N$. Find the equilibria of the ordinary differential equation and determine whether they are stable or unstable.

6. (8 points) Use the Laplace transform to find the solution to the following initial value problem.

$$\begin{aligned} y''(t) + 4y(t) &= 4 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

7. (8 points) The motion of a mass attached to a damped spring is described by the following initial value problem.

$$\begin{aligned} u''(t) + 4u'(t) + 4u(t) &= 36\cos 6t \\ u(0) &= 0 \\ u'(0) &= 0 \end{aligned}$$

Find the steady-state solution to this IVP.

8. (9 points) Find the general solution to the following ordinary differential equation.

$$y'' + y = \tan t$$

9. (8 points) Find the integrating factor which makes the following ordinary differential equation exact.

$$y\,dt + (3 + 3t - y)\,dy = 0$$

You do not need to solve the differential equation.

10. (8 points) The ordinary differential equation

$$4x\,y'' + 2y' + y = 0$$

has a regular singular point at $x_0 = 0$. One of the exponents of singularity is $r = 1/2$. In this case the recurrence relation for a series solution of the form $y(x) = x^{1/2}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$ is

$$a_{n+1} = -\frac{a_n}{(2n+2)(2n+3)} \quad \text{for } n \geq 0.$$

Find the first four non-zero coefficients in this series solution.

11. (9 points) Find the general solution to the following ordinary differential equation.

$$(y^2 e^y + 2t)\, dt + te^y (y^2 + 2y)\, dy = 0$$

12. (8 points) A mathematical model for the motion of a pendulum is

$$\frac{d^2\theta}{dt^2} - \frac{\gamma}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0$$

where m is the mass of the pendulum, γ is the coefficient of damping, g is the gravitational acceleration constant, l is the length of the pendulum arm, t is time, and θ is the angle the pendulum are makes with the downward oriented vertical. For small angles $\sin\theta \approx \theta$. Use this approximation to find an approximation to the motion of the pendulum.