Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Final Examination
April 28, 2009, 10:15AM–12:15PM

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (9 points) Solve the following initial value problem.

$$\begin{aligned} y' + 2t\,y &= 2t \\ y(0) &= 1 \end{aligned}$$

The ordinary differential equation is of the first-order linear type and thus we must find an integrating factor in order to solve it.

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}$$

Multiplying both sides of the ODE by $\mu(t)$ yields

$$\begin{aligned} e^{t^2}\,(y' + 2t\,y) &= 2t\,e^{t^2} \\ \left(e^{t^2} y\right) &= 2t\,e^{t^2} \\ \int \left(e^{t^2} y\right)\,dt &= \int 2t\,e^{t^2}\,dt \\ e^{t^2} y(t) &= e^{t^2} + C \\ y(t) &= 1 + Ce^{-t^2} \end{aligned}$$

Since $y(0) = 1$ then $C = 0$ and the solution to the initial value problem is

$$y(t) = 1.$$

2. (9 points) A hard-boiled egg is removed from a pot of hot water. Initially, the egg's temperature is 180°F. After 15 minutes its temperature is 140°F. If the environment's temperature is 65°F, when will the egg have a temperature of 100°F?

   Newton's Law of Cooling states that

$$y'(t) = k(y(t) - T_a)$$

   where $k < 0$ is a constant and T_a is the temperature of the environment. Thus we must solve the following initial value problem which is of the separable first-order type.

$$\begin{aligned} y'(t) &= k(y(t) - 65) \\ y(0) &= 180 \end{aligned}$$

   Separating the variables and integrating yields

$$\begin{aligned} \frac{1}{y(t) - 65}\,dy &= k\,dt \\ \int \frac{1}{y(t) - 65}\,dy &= \int k\,dt \\ \ln|y(t) - 65| &= kt + C \\ y(t) - 65 &= e^{kt+C} \\ y(t) &= 65 + Ce^{kt} \end{aligned}$$

   Since $y(0) = 180$ then $C = 115$. Since $y(15) = 140$ we can solve for k.

$$\begin{aligned} 140 &= 65 + 115e^{15k} \\ 75 &= 115e^{15k} \\ \frac{15}{23} &= e^{15k} \\ k &= \frac{1}{15}\ln\left(\frac{15}{23}\right) \approx -0.028496 \end{aligned}$$

   Thus the egg will reach a temperature of 100°F when

$$\begin{aligned} 100 &= 65 + 115e^{\frac{t}{15}\ln\left(\frac{15}{23}\right)} \\ 35 &= 115e^{\frac{t}{15}\ln\left(\frac{15}{23}\right)} \\ \frac{7}{23} &= e^{\frac{t}{15}\ln\left(\frac{15}{23}\right)} \\ \ln\left(\frac{7}{23}\right) &= \frac{t}{15}\ln\left(\frac{15}{23}\right) \\ t &= \frac{15\ln\left(\frac{7}{23}\right)}{\ln\left(\frac{15}{23}\right)} \approx 42 \quad \text{min.} \end{aligned}$$

3. (8 points) The value $x_0 = 0$ is an ordinary point for the ordinary differential equation below. Find the recurrence relation for a series solution to the ODE centered at $x_0 = 0$.

$$(1 + x^2)y'' + y = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} a_n x^n$ then by differentiating this solution and substituting it into the ODE we obtain

$$\begin{aligned}
0 &= (1 + x^2) \sum_{n=2} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2} n(n-1)a_n x^{n-2} + \sum_{n=2} n(n-1)a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0} (n+1)(n+2)a_{n+2} x^n + \sum_{n=0} n(n-1)a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} \Big[(n+1)(n+2)a_{n+2} + (n^2 - n + 1)a_n\Big] x^n
\end{aligned}$$

Equating coefficients in the power series on both sides of the equations produces a recurrence relation.

$$a_{n+2} = -\frac{(n^2 - n + 1)a_n}{(n+1)(n+2)} \quad \text{for } n \geq 0.$$

4. (9 points) Solve the following initial value problem.

$$\begin{aligned} y'' - y' - 2y &= 2\sin 2t \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

The complementary solution can be found by solving the homogeneous ODE

$$\begin{aligned} y'' - y' - 2y &= 0 \\ r^2 - r - 2 &= 0 \\ (r-2)(r+1) &= 0 \end{aligned}$$

Thus the roots of the characteristic polynomial are $r_1 = -1$ and $r_2 = 2$. The complementary solution therefore has the form $y_c(t) = c_1 e^{-t} + c_2 e^{2t}$. We will use the method of undetermined coefficients to find a particular solution. Assuming $Y(t) = A\cos 2t + B\sin 2t$, differentiating and substituting into the ODE we obtain

$$\begin{aligned} 2\sin 2t &= -4A\cos 2t - 4B\sin 2t - (-2A\sin 2t + 2B\cos 2t) - 2(A\cos 2t + B\sin 2t) \\ &= (-6A - 2B)\cos 2t + (2A - 6B)\sin 2t \end{aligned}$$

Thus we must solve the following system of two equations in the two unknowns A and B.

$$\begin{aligned} 6A + 2B &= 0 \\ 2A - 6B &= 2 \end{aligned}$$

Multiplying the first equation by 3 and adding to the second equation we find that $A = 1/10$ and thus that $B = -3/10$. Consequently the general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{2t} + \frac{1}{10}\cos 2t - \frac{3}{10}\sin 2t.$$

We may use the initial conditions to evaluate the constants.

$$\begin{aligned} y(0) = 0 &= c_1 + c_2 + \frac{1}{10} \\ y'(0) = 1 &= -c_1 + 2c_2 - \frac{6}{10} \end{aligned}$$

Adding the two equations together we see that $c_2 = 1/2$ and thus that $c_1 = -3/5$. The solution to the IVP is

$$y(t) = -\frac{3}{5}e^{-t} + \frac{1}{2}e^{2t} + \frac{1}{10}\cos 2t - \frac{3}{10}\sin 2t.$$

5. (8 points) Suppose a population of birds obeys the logistic equation with threshold given below.

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right)P$$

where $k > 0$ and $0 < M < N$. Find the equilibria of the ordinary differential equation and determine whether they are stable or unstable.

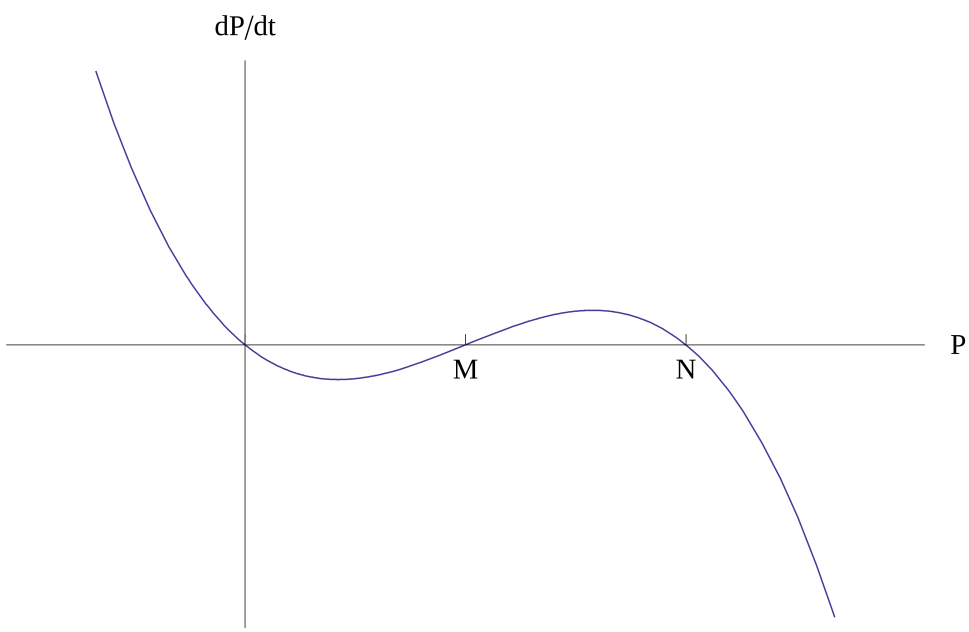
The equilibria are values of P for which $\frac{dP}{dt} = 0$.

$$0 = k\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right)P$$

implies the equilibria are the values

$$\begin{aligned} P_1 &= 0 \\ P_2 &= M \\ P_3 &= N \end{aligned}$$

Note that $P_1 < P_2 < P_3$. If we plot $\frac{dP}{dt}$ *versus* P the graph resembles the following



Thus $P_1 = 0$ is stable, $P_2 = M$ is unstable, and $P_3 = N$ is stable.

6. (8 points) Use the Laplace transform to find the solution to the following initial value problem.

$$\begin{aligned} y''(t) + 4y(t) &= 4 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

Taking the Laplace transform of both sides of the ODE yields

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{4}{s} \\ (s^2+4)Y(s) - s &= \frac{4}{s} \\ (s^2+4)Y(s) &= s + \frac{4}{s} \\ Y(s) &= \frac{s}{s^2+4} + \frac{4}{s(s^2+4)} \\ &= \frac{s^2}{s(s^2+4)} + \frac{4}{s(s^2+4)} \\ &= \frac{s^2+4}{s(s^2+4)} \\ &= \frac{1}{s} \\ y(t) &= 1. \end{aligned}$$

7. (8 points) The motion of a mass attached to a damped spring is described by the following initial value problem.

$$\begin{aligned} u''(t) + 4u'(t) + 4u(t) &= 36\cos 6t \\ u(0) &= 0 \\ u'(0) &= 0 \end{aligned}$$

Find the steady-state solution to this IVP.

The steady-state solution will be the particular solution for the nonhomogeneous equation. We will use the method of undetermined coefficients and assume that $U(t) = A\cos 6t + B\sin 6t$. Differentiating this function and substituting into the ODE produces

$$\begin{aligned} 36\cos 6t &= -36A\cos 6t - 36B\sin 6t + 4(-6A\sin 6t + 6B\cos 6t) + 4(A\cos 6t + B\sin 6t) \\ &= (-32A + 24B)\cos 6t + (-24A - 32B)\sin 6t. \end{aligned}$$

Equating coefficients on both sides of the equation implies the following two equations in the unknowns A and B.

$$\begin{aligned} 36 &= -32A + 24B \\ 0 &= 24A + 32B \end{aligned}$$

The second equation suggests that $B = -\frac{3}{4}A$. Substituting this into the first equation enables us to see that

$$36 = -32A - 18A = -50A \quad \Longrightarrow \quad A = -\frac{18}{25}$$

and thus that $B = \frac{27}{50}$. Consequently the steady-state solution is

$$U(t) = -\frac{18}{25}\cos 6t + \frac{27}{50}\sin 6t.$$

8. (9 points) Find the general solution to the following ordinary differential equation.

$$y'' + y = \tan t$$

Two solutions to the homogeneous equation $y'' + y = 0$ are $y_1(t) = \cos t$ and $y_2(t) = \sin t$. Their Wronskian is $W(t) = 1$. Using the method of variation of parameters we see that

$$\begin{aligned}
\mu_1'(t) &= -\frac{\sin t\ \tan t}{1} \\
&= -\frac{\sin^2 t}{\cos t} \\
&= \frac{\cos^2 t - 1}{\cos t} \\
&= \cos t - \sec t \\
\mu_1(t) &= \sin t - \ln|\sec t + \tan t| \\
\mu_2'(t) &= \frac{\cos t\ \tan t}{1} \\
&= \sin t \\
\mu_2(t) &= -\cos t.
\end{aligned}$$

Thus the general solution to the ODE is

$$\begin{aligned}
y(t) &= c_1 \cos t + c_2 \sin t + (\sin t - \ln|\sec t + \tan t|)\cos t - \cos t\ \sin t \\
&= c_1 \cos t + c_2 \sin t - (\cos t)\ln|\sec t + \tan t|.
\end{aligned}$$

9. (8 points) Find the integrating factor which makes the following ordinary differential equation exact.

$$y\,dt + (3 + 3t - y)\,dy = 0$$

You do not need to solve the differential equation.

Assuming the integrating factor is of the form $\mu(y)$ then multiplying the ODE by this factor yields

$$\mu(y)y\,dt + \mu(y)(3 + 3t - y)\,dy = 0$$

This equation is exact if

$$\begin{aligned}
\frac{\partial}{\partial y}(\mu(y)y) &= \frac{\partial}{\partial t}(\mu(y)(3 + 3t - y)) \\
\mu'(y)y + \mu(y) &= 3\mu(y) \\
\mu'(y)y - 2\mu(y) &= 0 \\
\mu'(y) - \frac{2}{y}\mu(y) &= 0.
\end{aligned}$$

This first order linear ordinary differential equation can be solved by multiplying it by its own integrating factor

$$\eta(y) = e^{-\int \frac{2}{y}\,dy} = e^{-2\ln y} = \frac{1}{y^2}.$$

Thus we have

$$\begin{aligned}
\mu'(y) - \frac{2}{y}\mu(y) &= 0 \\
\frac{1}{y^2}(\mu'(y) - \frac{2}{y}\mu(y)) &= 0 \\
\frac{d}{dy}(\frac{1}{y^2}\mu(y)) &= 0 \\
\frac{1}{y^2}\mu(y) &= C \\
\mu(y) &= Cy^2.
\end{aligned}$$

If we choose $C = 1$ then the original ODE is made exact when we multiply by $\mu(y) = y^2$.

10. (8 points) The ordinary differential equation

$$4x\, y'' + 2y' + y = 0$$

has a regular singular point at $x_0 = 0$. One of the exponents of singularity is $r = 1/2$. In this case the recurrence relation for a series solution of the form $y(x) = x^{1/2}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$ is

$$a_{n+1} = -\frac{a_n}{(2n+2)(2n+3)} \quad \text{for } n \geq 0.$$

Find the first four non-zero coefficients in this series solution.

Since $a_0 = 1$ then

$$\begin{aligned} a_1 &= -\frac{1}{(2)(3)} = -\frac{1}{3!} \\ a_2 &= -\frac{a_1}{(4)(5)} = \frac{1}{5!} \\ a_3 &= -\frac{a_2}{(6)(7)} = -\frac{1}{7!} \end{aligned}$$

11. (9 points) Find the general solution to the following ordinary differential equation.

$$(y^2e^y + 2t)\,dt + te^y(y^2 + 2y)\,dy = 0$$

Since

$$\frac{\partial}{\partial y}(y^2e^y + 2t) = 2ye^y + y^2e^y = \frac{\partial}{\partial t}[te^y(y^2 + 2y)]$$

the equation is exact.

$$\begin{aligned}\Psi(t, y) &= \int (y^2e^y + 2t)\,dt \\ &= ty^2e^y + t^2 + h(y)\end{aligned}$$

where $h(y)$ is an arbitrary function of y.

$$\begin{aligned}\frac{\partial}{\partial y}(ty^2e^y + t^2 + h(y)) &= te^y(y^2 + 2y) \\ 2tye^y + ty^2e^y + h'(y) &= te^y(y^2 + 2y) \\ h'(y) &= 0\end{aligned}$$

Thus $h(y)$ is a constant and the implicit form of the solution to the ODE is

$$ty^2e^y + t^2 = C$$

where C is a constant.

12. (8 points) A mathematical model for the motion of a pendulum is

$$\frac{d^2\theta}{dt^2} - \frac{\gamma}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0$$

where m is the mass of the pendulum, γ is the coefficient of damping, g is the gravitational acceleration constant, l is the length of the pendulum arm, t is time, and θ is the angle the pendulum are makes with the downward oriented vertical. For small angles $\sin\theta \approx \theta$. Use this approximation to find an approximation to the motion of the pendulum.

Replacing $\sin\theta$ with θ yields the second order linear constant coefficient homogeneous equation

$$\frac{d^2\theta}{dt^2} - \frac{\gamma}{m}\frac{d\theta}{dt} + \frac{g}{l}\theta = 0$$

whose characteristic equation is

$$\begin{aligned} r^2 - \frac{\gamma}{m}r + \frac{g}{l} &= 0 \\ r &= \frac{\frac{\gamma}{m} \pm \sqrt{\left(-\frac{\gamma}{m}\right)^2 - 4\frac{g}{l}}}{2} \\ &= \frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{g}{l}}. \end{aligned}$$

If we assume the pendulum is underdamped then

$$\frac{\gamma^2}{4m^2} - \frac{g}{l} < 0$$

and the solution to the ODE takes on the form

$$\theta(t) = e^{\gamma t/2m}\left(\cos\sqrt{\frac{g}{l} - \frac{\gamma^2}{4m^2}}t + \sin\sqrt{\frac{g}{l} - \frac{\gamma^2}{4m^2}}t\right).$$