Millersville University **Name** ______________________________

Department of Mathematics

MATH 365, *Ordinary Differential Equations*, Test 1

February 11, 2009

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (12 points each) Solve each of the following ODEs or IVPs.

   (a) $\dfrac{dy}{dt} = t\,y^3(1+t^2)^{-1/2}, \quad y(0) = 1$

   Using the method of separation of the variables we obtain

$$
\begin{aligned}
\frac{dy}{dt} &= t\,y^3(1+t^2)^{-1/2} \\
\frac{1}{y^3}\,dy &= \frac{t}{\sqrt{1+t^2}}\,dt \\
\int_1^y \frac{1}{u^3}\,du &= \int_0^t \frac{s}{\sqrt{1+s^2}}\,ds \\
\left.\frac{-1}{2u^2}\right|_1^y &= \left.\sqrt{1+s^2}\right|_0^t \\
\frac{1}{2} - \frac{1}{2y^2} &= \sqrt{1+t^2} - 1 \\
\frac{3}{2} - \frac{1}{2y^2} &= \sqrt{1+t^2} \\
\frac{3}{2} - \sqrt{1+t^2} &= \frac{1}{2y^2} \\
3 - 2\sqrt{1+t^2} &= \frac{1}{y^2} \\
y^2 &= \frac{1}{3 - 2\sqrt{1+t^2}} \\
y &= \frac{1}{\sqrt{3 - 2\sqrt{1+t^2}}}
\end{aligned}
$$

   (b) $t\dfrac{dy}{dt} - y = t^2 e^{-t}$

   We may re-write the ODE in the form

$$\frac{dy}{dt} - \frac{1}{t}\,y = te^{-t}$$

which is a first-order linear ODE. An integrating factor for this equation is

$$\mu(t) = e^{-\int \frac{1}{t}\,dt} = e^{-\ln t} = \frac{1}{t}.$$

Multiplying both sides of the ODE by the integrating factor produces

$$\begin{aligned}
\frac{1}{t}\left[\frac{dy}{dt} - \frac{1}{t}\,y\right] &= e^{-t} \\
\frac{d}{dt}\left[\frac{1}{t}\,y\right] &= e^{-t} \\
\frac{y}{t} &= -e^{-t} + C \\
y(t) &= -t\,e^{-t} + Ct
\end{aligned}$$

(c) $\dfrac{dy}{dt} = -\dfrac{3t + ty^2}{2y + t^2y}$

Using the method of separation of the variables we obtain

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{3t + ty^2}{2y + t^2y} \\
&= -\frac{t(3 + y^2)}{y(2 + t^2)} \\
\frac{y}{y^2+3}\,dy &= -\frac{t}{t^2+2}\,dt \\
\int \frac{y}{y^2+3}\,dy &= -\int \frac{t}{t^2+2}\,dt \\
\frac{1}{2}\ln(y^2+3) &= -\frac{1}{2}\ln(t^2+2) + C \\
\ln\sqrt{y^2+3} &= -\ln\sqrt{t^2+2} + C \\
\sqrt{y^2+3} &= \frac{C}{\sqrt{t^2+2}} \\
y^2 + 3 &= \frac{C}{t^2+2} \\
y^2 &= \frac{C}{t^2+2} - 3
\end{aligned}$$

(d) $(ty^2 - 1)\,dt + (t^2 y - 1)\,dy = 0$

Since

$$\frac{\partial}{\partial y}(ty^2 - 1) = 2ty = \frac{\partial}{\partial t}(t^2 y - 1)$$

the ODE is exact. The implicit form of the solution is

$$\begin{aligned} \psi(t, y) &= \int (ty^2 - 1)\,dy \\ &= \frac{t^2\,y^2}{2} - t + h(y) \end{aligned}$$

where $h(y)$ is an arbitrary function of y. We know that

$$\begin{aligned} \frac{\partial}{\partial y}\psi(t, y) &= \frac{\partial}{\partial y}\left(\frac{t^2\,y^2}{2} - t + h(y)\right) \\ t^2\,y - 1 &= t^2\,y + h'(y) \\ -1 &= h'(y) \\ -y &= h(y). \end{aligned}$$

Thus the implicit form of the solution is

$$\psi(t, y) = \frac{t^2\,y^2}{2} - t - y = C.$$

2. (12 points) A tank with a volume of 10 liters initially contains water and 20 grams of dissolved salt. A solution of water containing 10 grams of salt per liter flows into the tank at a rate of 2 liters per minute and a well-mixed solution of water and salt flows out of the tank at the same rate. Find the concentration of salt in the tank after 3 minutes.

Let $V = 10$ liters be the volume of the tank, $A(t)$ be the amount (in grams) of salt in the tank, and $C(t) = A(t)/V$ be the concentration of salt in the tank.

$$\begin{aligned}
\frac{dA}{dt} &= (2)(10) - (2)C(t) \\
\frac{1}{V}\frac{dA}{dt} &= \frac{1}{V}\left[(2)(10) - (2)C(t)\right] \\
\frac{dC}{dt} &= 2 - \frac{1}{5}C(t).
\end{aligned}$$

Thus the mixed-tank situation can be described by the initial value problem

$$\begin{aligned}
\frac{dC}{dt} + \frac{1}{5}C(t) &= 2 \\
C(0) &= \frac{20}{10} = 2.
\end{aligned}$$

Solving this first-order linear IVP yields

$$\begin{aligned}
\frac{dC}{dt} + \frac{1}{5}C(t) &= 2 \\
e^{t/5}\left[\frac{dC}{dt} + \frac{1}{5}C(t)\right] &= 2e^{t/5} \\
\frac{d}{dt}\left[e^{t/5}C(t)\right] &= 2e^{t/5} \\
e^{t/5}C(t) - C(0) &= 10e^{t/5} - 10 \\
e^{t/5}C(t) &= 10e^{t/5} - 10 + 2 \\
C(t) &= 10 - 8e^{-t/5}.
\end{aligned}$$

Therefore

$$C(3) = 10 - 8e^{-3/5} \approx 5.61 \quad \text{g/L}.$$

3. (10 points) Let $y(t) = c_1 t + c_2\left(1 + \frac{t}{2}\ln\frac{1-t}{1+t}\right)$ and show that y solves the ODE below. The expressions c_1 and c_2 are constants.

$$(1-t^2)y'' - 2ty' + 2y = 0.$$

Note that

$$\begin{aligned}
y'(t) &= c_1 + \frac{c_2}{2}\ln\frac{1-t}{1+t} + \frac{c_2 t}{2}\frac{1+t}{1-t}\frac{(-1)(1+t)-(1-t)}{(1+t)^2} \\
&= c_1 + \frac{c_2}{2}\ln\frac{1-t}{1+t} + \frac{c_2 t}{2}\frac{1}{1-t}\frac{(-1)(1+t)-(1-t)}{1+t} \\
&= c_1 + \frac{c_2}{2}\ln\frac{1-t}{1+t} + \frac{c_2 t}{2}\frac{1}{1-t}\frac{(-2)}{1+t} \\
&= c_1 + \frac{c_2}{2}\ln\frac{1-t}{1+t} - \frac{c_2 t}{1-t^2} \\
y''(t) &= \frac{c_2}{2}\frac{1+t}{1-t}\frac{(-1)(1+t)-(1-t)}{(1+t)^2} - c_2\frac{(1-t^2)-t(-2t)}{(1-t^2)^2} \\
&= \frac{c_2}{2}\frac{1}{1-t}\frac{(-1)(1+t)-(1-t)}{1+t} - c_2\frac{1+t^2}{(1-t^2)^2} \\
&= \frac{c_2}{2}\frac{1}{1-t}\frac{(-2)}{1+t} - c_2\frac{1+t^2}{(1-t^2)^2} \\
&= -\frac{c_2}{1-t^2} - c_2\frac{1+t^2}{(1-t^2)^2}.
\end{aligned}$$

Substituting into the ODE produces

$$\begin{aligned}
0 &= (1-t^2)\left[-\frac{c_2}{1-t^2} - c_2\frac{1+t^2}{(1-t^2)^2}\right] - 2t\left[c_1 + \frac{c_2}{2}\ln\frac{1-t}{1+t} - \frac{c_2 t}{1-t^2}\right] \\
&\quad + 2\left[c_1 t + c_2\left(1 + \frac{t}{2}\ln\frac{1-t}{1+t}\right)\right] \\
&= -c_2 - c_2\frac{1+t^2}{1-t^2} - 2c_1 t - c_2 t\ln\frac{1-t}{1+t} + c_2\frac{2t^2}{1-t^2} + 2c_1 t + 2c_2 + c_2 t\ln\frac{1-t}{1+t} \\
&= c_2 - c_2\frac{1+t^2}{1-t^2} + c_2\frac{2t^2}{1-t^2} \\
&= c_2\frac{(1-t^2)-(1+t^2)+2t^2}{1-t^2} \\
&= 0.
\end{aligned}$$

Thus $y(t)$ is a solution to the ODE.

4. (14 points) Use the method of successive approximations to find the first four terms in the sequence of successive approximations to the solution of the initial value problem:

$$\begin{aligned} \frac{dy}{dt} &= -y+1 \\ y(0) &= 0. \end{aligned}$$

$$\begin{aligned} \phi_0(t) &= 0 \\ \phi_1(t) &= \int_0^t (-0+1)\,ds \\ &= t \\ \phi_2(t) &= \int_0^t (-s+1)\,ds \\ &= \left(-\frac{s^2}{2}+s\right)\Bigg|_0^t \\ &= -\frac{t^2}{2}+t \\ \phi_3(t) &= \int_0^t \left(-\left[-\frac{s^2}{2}+s\right]+1\right)\,ds \\ &= \int_0^t \left(\frac{s^2}{2}-s+1\right)\,ds \\ &= \left(\frac{s^3}{6}-\frac{s^2}{2}+s\right)\Bigg|_0^t \\ &= \frac{t^3}{6}-\frac{t^2}{2}+t \end{aligned}$$

5. (8 points each) The Schaefer model of a population of fish subject to harvesting by commercial fishing is

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right) - Ey$$

(a) Suppose that r, K, and E are positive constants and $E < r$. Determine all the equilibria for the Schaefer model.

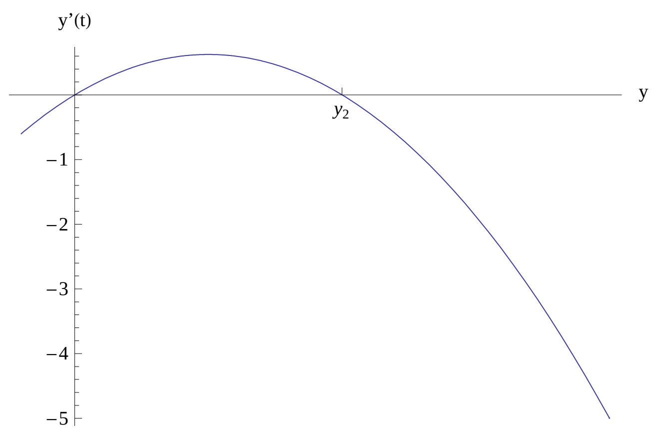
At equilibrium $dy/dt = 0$ and thus

$$\begin{aligned} 0 &= ry\left(1 - \frac{y}{K}\right) - Ey \\ &= y\left[r\left(1 - \frac{y}{K}\right) - E\right]. \end{aligned}$$

The two equilibria are $y_1 = 0$ and $y_2 = K\left(1 - \frac{E}{r}\right) > 0$.

(b) Classify each equilibrium as stable or unstable.

If we plot dy/dt versus y we have the following.



Thus y_1 is unstable and y_2 is stable.