

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (12 points) Find the Wronskian of two solutions of Legendre's equation given below.

$$(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0$$

Re-writing the ODE in the form

$$y'' - \frac{2t}{1 - t^2}y' + \frac{\alpha(\alpha + 1)}{1 - t^2}y = 0$$

Abel's theorem implies the Wronskian of two solutions is a function of the form

$$\begin{aligned} W(t) &= Ce^{-\int \left(-\frac{2t}{1-t^2}\right) dt} \\ &= Ce^{\int \frac{2t}{1-t^2} dt} \\ &= Ce^{-\ln|1-t^2|} \\ &= \frac{C}{|1-t^2|}, \end{aligned}$$

where  $C$  is a constant.

2. (6 points) A spring-mass system is setup with a mass of 5 kg which stretched the spring 0.20 m. If the system is critically damped, what is the coefficient of damping  $\gamma$ ?

Let  $m = 5$  and note that

$$\begin{aligned} mg &= kL \\ 5(9.8) &= k(0.20) \\ k &= 245 \text{ m/N}. \end{aligned}$$

Critical damping occurs when

$$\gamma = 2\sqrt{km} = 2\sqrt{(245)(5)} = 70 \text{ N/m/s}.$$

3. (12 points) Find the general solution to the ordinary differential equation below.

$$9y'' + 9y' - 4y = 0$$

The characteristic equation of this second order linear constant coefficient homogeneous ODE is

$$\begin{aligned} 0 &= 9r^2 + 9r - 4 \\ &= (3r + 4)(3r - 1) \\ r_1 &= -\frac{4}{3} \\ r_2 &= \frac{1}{3}. \end{aligned}$$

Thus the general solution to the ODE is

$$y(t) = c_1 e^{-4t/3} + c_2 e^{t/3}$$

where  $c_1$  and  $c_2$  are constants.

4. (16 points) Find the general solution to the ordinary differential equation below.

$$y'' + 9y = 4 \csc 3t$$

You may assume  $0 < t < \pi/3$ .

The complementary solution to the ODE can be found by solving the homogeneous equation

$$y'' + 9y = 0$$

whose characteristic roots are  $r = 0 \pm 3i$  where  $i = \sqrt{-1}$ . Thus  $y_c(t) = c_1 \cos 3t + c_2 \sin 3t$  where  $c_1$  and  $c_2$  are constants. Letting  $y_1(t) = \cos 3t$  and  $y_2(t) = \sin 3t$ , the Wronskian of  $y_1$  and  $y_2$  is

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos 3t & \sin 3t \\ -3 \sin 3t & 3 \cos 3t \end{vmatrix} = 3 \cos^2 3t + 3 \sin^2 3t = 3.$$

Applying the variation of parameters formula produces

$$\begin{aligned} \mu_1'(t) &= -\frac{4 \csc 3t \sin 3t}{3} \\ &= -\frac{4}{3} \\ \mu_1(t) &= -\frac{4}{3}t \\ \mu_2'(t) &= \frac{4 \csc 3t \cos 3t}{3} \\ &= \frac{4}{3} \cot 3t \\ \mu_2(t) &= \frac{4}{9} \ln(\sin 3t). \end{aligned}$$

Therefore the general solution to the nonhomogeneous ODE is

$$y(t) = c_1 \cos 3t + c_2 \sin 3t - \frac{4}{3}t \cos 3t + \frac{4}{9} \ln(\sin 3t) \sin 3t.$$

5. (14 points) Find the solution to the initial value problem below.

$$\begin{aligned}y'' + 2y' + 2y &= 0 \\y(\pi/4) &= 0 \\y'(\pi/4) &= -1\end{aligned}$$

The characteristic equation for this ODE is

$$\begin{aligned}0 &= r^2 + 2r + 2 \\r &= \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = -1 \pm i\end{aligned}$$

where  $i = \sqrt{-1}$ . The general solution is then of the form

$$\begin{aligned}y(t) &= e^{-t}(c_1 \cos t + c_2 \sin t) \\y'(t) &= e^{-t}((-c_1 + c_2) \cos t + (-c_1 - c_2) \sin t).\end{aligned}$$

Applying the initial conditions produces the system of equations:

$$\begin{aligned}0 &= e^{-\pi/4}(c_1 \cos \frac{\pi}{4} + c_2 \sin \frac{\pi}{4}) \\-1 &= e^{-\pi/4}\left((-c_1 + c_2) \cos \frac{\pi}{4} + (-c_1 - c_2) \sin \frac{\pi}{4}\right)\end{aligned}$$

The first equation is equivalent to  $0 = c_1 + c_2$  which implies  $c_2 = -c_1$ . Substituting this into the second equation produces

$$\begin{aligned}-1 &= e^{-\pi/4}\left(-2c_1 \frac{\sqrt{2}}{2}\right) \\c_1 &= \frac{1}{\sqrt{2}}e^{\pi/4}\end{aligned}$$

and  $c_2 = -\frac{1}{\sqrt{2}}e^{\pi/4}$ . Thus the solution to the initial value problem is

$$y(t) = \frac{1}{\sqrt{2}}e^{-t+\pi/4}(\cos t - \sin t).$$

6. (12 points) Consider the ordinary differential equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 0$$

for which  $y_1(t) = t$  is a solution. Use the method of reduction of order to find a second linearly independent solution to the ODE.

Suppose  $y_2(t) = tv(t)$ , then if  $y_2(t)$  solves the ODE

$$\begin{aligned} 0 &= t^2(tv(t))'' - t(t+2)(tv(t))' + (t+2)tv(t) \\ &= t^2(2v'(t) + tv''(t)) - t(t+2)(v(t) + tv'(t)) + (t+2)tv(t) \\ &= t^3v''(t) + (2t^2 - t^2(t+2))v'(t) + (-t(t+2) + t(t+2))v(t) \\ &= t^3v''(t) - t^3v'(t) \\ 0 &= v''(t) - v'(t) \\ v'(t) &= e^t \\ v(t) &= e^t. \end{aligned}$$

A second, linearly independent solution to the ODE is  $y_2(t) = te^t$ .

7. (16 points) Find the general solution to the ordinary differential equation below.

$$y'' + 2y' + y = 2t + \sin(2t)$$

The complementary solution can be found by solving the homogeneous equation

$$y'' + 2y' + y = 0$$

whose characteristic equation is

$$0 = r^2 + 2r + 1 = (r + 1)^2 \quad \text{thus} \quad r_1 = r_2 = -1.$$

The complementary solution is therefore  $y_c(t) = e^{-t}(c_1 + c_2t)$  where  $c_1$  and  $c_2$  are constants.

The particular solution can be found using the method of undetermined coefficients. If  $Y(t) = At + B + C \sin(2t) + D \cos(2t)$  then

$$\begin{aligned} 2t + \sin(2t) &= (At + B + C \sin(2t) + D \cos(2t))'' + 2(At + B + C \sin(2t) + D \cos(2t))' \\ &\quad + At + B + C \sin(2t) + D \cos(2t) \\ &= -4C \sin(2t) - 4D \cos(2t) + 2(A + 2C \cos(2t) - 2D \sin(2t)) \\ &\quad + At + B + C \sin(2t) + D \cos(2t) \\ &= (-3C - 4D) \sin(2t) + (-3D + 4C) \cos(2t) + At + (2A + B). \end{aligned}$$

Equating the coefficients of  $t$  on both sides of the equation reveals that  $A = 2$ . Equating the constants on both sides of the equation yields  $B = -4$ . The remaining two equations for unknowns  $C$  and  $D$  are

$$\begin{aligned} 1 &= -3C - 4D \\ 0 &= 4C - 3D. \end{aligned}$$

Multiplying the first equation by 3 and the second equation by  $-4$  and adding the results produces  $C = -3/25$  and  $D = -4/25$ . Thus the general solution to the nonhomogeneous ODE is

$$y(t) = e^{-t}(c_1 + c_2t) + 2t - 4 - \frac{3}{25} \sin(2t) - \frac{4}{25} \cos(2t).$$

8. (6 points each) A spring-mass system is described by the ordinary differential equation

$$u'' + \gamma u' + u = 0.$$

(a) What is the period of the undamped solution to this ODE?

If  $\gamma = 0$ ,  $m = 1$ , and  $k = 1$  then

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi.$$

(b) Find the value of  $\gamma$  for which the quasi period of the damped system is  $3\pi$ .

$$\begin{aligned} T_d &= \frac{2\pi}{\mu} \\ &= \frac{2\pi}{\frac{\sqrt{4mk - \gamma^2}}{2m}} \\ 3\pi &= \frac{2\pi}{\frac{\sqrt{4 - \gamma^2}}{2}} \\ 3 &= \frac{4}{\sqrt{4 - \gamma^2}} \\ \frac{4}{3} &= \sqrt{4 - \gamma^2} \\ \frac{16}{9} &= 4 - \gamma^2 \\ \frac{20}{9} &= \gamma^2 \\ \gamma &= \frac{2\sqrt{5}}{3} \approx 1.49071 \end{aligned}$$