Millersville University **Name** **Answer Key**

Department of Mathematics

MATH 365, *Ordinary Differential Equations*, Test 3

April 17, 2009

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (18 points) Find the recurrence relation for a power series solution to

$$(1-x)y'' - xy' + y = 0$$

near the ordinary point $x_0 = 0$.

Assuming $y(x) = \sum_{n=0}^{\infty} a_n x^n$ then differentiating this solution and substituting into the ODE produces

$$\begin{aligned}
0 &= (1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}(n-1)a_n x^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}(n-1)a_n x^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}(n-1)a_n x^n
\end{aligned}$$

Thus the recurrence relation is

$$a_{n+2} = \frac{n(n+1)a_{n+1} + (n-1)a_n}{(n+1)(n+2)} \quad \text{for } n \geq 0.$$

2. (16 points) For the following ordinary differential equation, $x_0 = 0$ is a regular singular point. Find the exponents of singularity corresponding to this regular singular point. You do not need to solve the ODE.

$$x^2 y'' + 3(\sin x)y' + y = 0$$

$$\begin{aligned} p_0 &= \lim_{x \to 0} x \left(\frac{3 \sin x}{x^2} \right) = 3 \lim_{x \to 0} \frac{\sin x}{x} = 3 \\ q_0 &= \lim_{x \to 0} x^2 \left(\frac{1}{x^2} \right) = 1 \end{aligned}$$

Thus the indicial equation is

$$\begin{aligned} 0 &= r(r-1) + 3r + 1 \\ &= r^2 + 2r + 1 \\ &= (r+1)^2. \end{aligned}$$

Thus the exponents of singularity are $r_1 = r_2 = -1$.

3. (16 points) For the following ordinary differential equation find all the singular points and determine which are regular and which are irregular. You do not need to solve the ODE.

$$(\cos x)y'' + \left(\frac{1}{x - \frac{\pi}{2}}\right) y' + y = 0$$

The singular points are the values of x for which $\cos x = 0$. Thus the singular points are of the form $x_0 = \dfrac{(2n+1)\pi}{2}$ for $n = 0, \pm 1, \pm 2, \ldots$.

$$\lim_{x \to \frac{(2n+1)\pi}{2}} \left(x - \frac{(2n+1)\pi}{2}\right)\left(\frac{\frac{1}{x-\frac{\pi}{2}}}{\cos x}\right) = \begin{cases} \lim_{x\to\pi/2} \frac{1}{\cos x} & \text{if } n = 0, \\ \lim_{x \to \frac{(2n+1)\pi}{2}} \left(\frac{1}{x-\frac{\pi}{2}}\right) \frac{x - \frac{(2n+1)\pi}{2}}{\cos x} & \text{if } n \neq 0 \end{cases}$$
$$= \begin{cases} \text{Limit does not exist.} & \text{if } n = 0, \\ \frac{(-1)^{n+1}}{n\pi} & \text{if } n \neq 0. \end{cases}$$

We can also see that

$$\lim_{x \to \frac{(2n+1)\pi}{2}} \left(x - \frac{(2n+1)\pi}{2}\right)^2 \left(\frac{1}{\cos x}\right) = \lim_{x \to \frac{(2n+1)\pi}{2}} \frac{\left(x - \frac{(2n+1)\pi}{2}\right)^2}{\cos x} = 0.$$

Thus $x_0 = \pi/2$ is an irregular singular point and $x_0 = (2n+1)\pi/2$ is a regular singular point for $n \neq 0$.

4. (16 points) Solve the following initial value problem.

$$\begin{aligned} x^2y'' - 3xy' + 4y &= 0 \\ y(1) &= 2 \\ y'(1) &= -3 \end{aligned}$$

If we let $x = e^z$ then the ordinary differential equation becomes

$$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 0$$

which is a constant coefficient second order linear homogeneous ordinary differential equation with characteristic equation

$$0 = r^2 - 4r + 4 = (r-2)^2.$$

Thus $r_1 = r_2 = r = 2$. The general solution to the ODE is

$$y(x) = (c_1 + c_2 \ln x)x^2.$$

Using the initial conditions we can evaluate the constants c_1 and c_2.

$$y(1) = 2 = c_1$$

Therefore

$$y'(x) = \frac{d}{dx}\left[(2 + c_2 \ln x)x^2\right] = c_2 x + (2 + c_2 \ln x)(2x).$$

Consequently

$$y'(1) = -3 = c_2 + 4 \quad \Longrightarrow \quad c_2 = -7 \quad \text{and}$$

the solution to the IVP is

$$y(x) = (2 - 7 \ln x)x^2.$$

5. (17 points) For the ordinary differential equation

$$(2x^2 + 4x)y'' + y' - xy = 0$$

$x_0 = 0$ is a regular singular point with indicial equation

$$0 = r^2 - \frac{3}{4}r.$$

Find the recurrence relation for a series solution centered at $x_0 = 0$ to this ODE corresponding to the larger of the two exponents of singularity.

The exponents of singularity are $r_2 = 0$ and $r_1 = 3/4$, thus we will search for a series solution of the form $y(x) = \sum_{n=0} a_n x^{n+3/4}$ with $a_0 \neq 0$. Differentiating this solution and substituting it into the ODE yields

$$\begin{aligned}
0 &= (2x^2+4x)\sum_{n=0}^{\infty}(n+\frac{3}{4})(n-\frac{1}{4})a_n x^{n-\frac{5}{4}} + \sum_{n=0}^{\infty}(n+\frac{3}{4})a_n x^{n-\frac{1}{4}} - x\sum_{n=0}^{\infty} a_n x^{n+\frac{3}{4}} \\
&= \sum_{n=0}^{\infty} 2(n+\frac{3}{4})(n-\frac{1}{4})a_n x^{n+\frac{3}{4}} + \sum_{n=0}^{\infty} 4(n+\frac{3}{4})(n-\frac{1}{4})a_n x^{n-\frac{1}{4}} + \sum_{n=0}^{\infty}(n+\frac{3}{4})a_n x^{n-\frac{1}{4}} \\
&\quad - \sum_{n=0}^{\infty} a_n x^{n+\frac{7}{4}} \\
&= \sum_{n=1}^{\infty} 2(n-\frac{1}{4})(n-\frac{5}{4})a_{n-1}x^{n-\frac{1}{4}} + \sum_{n=0}^{\infty}\left[4(n-\frac{1}{4})+1\right](n+\frac{3}{4})a_n x^{n-\frac{1}{4}} - \sum_{n=2}^{\infty} a_{n-2}x^{n-\frac{1}{4}} \\
&= \sum_{n=1}^{\infty} 2(n-\frac{1}{4})(n-\frac{5}{4})a_{n-1}x^{n-\frac{1}{4}} + \sum_{n=0}^{\infty} 4n(n+\frac{3}{4})a_n x^{n-\frac{1}{4}} - \sum_{n=2}^{\infty} a_{n-2}x^{n-\frac{1}{4}} \\
&= \sum_{n=1}^{\infty} 2(n-\frac{1}{4})(n-\frac{5}{4})a_{n-1}x^{n-\frac{1}{4}} + \sum_{n=1}^{\infty} 4n(n+\frac{3}{4})a_n x^{n-\frac{1}{4}} - \sum_{n=2}^{\infty} a_{n-2}x^{n-\frac{1}{4}} \\
&= \sum_{n=1}^{\infty}\left[2(n-\frac{1}{4})(n-\frac{5}{4})a_{n-1} + 4n(n+\frac{3}{4})a_n\right]x^{n-\frac{1}{4}} - \sum_{n=2}^{\infty} a_{n-2}x^{n-\frac{1}{4}} \\
&= \left(-\frac{3}{8}a_0 + 7a_1\right)x^{\frac{3}{4}} \\
&\quad + \sum_{n=2}^{\infty}\left[2(n-\frac{1}{4})(n-\frac{5}{4})a_{n-1} + 4n(n+\frac{3}{4})a_n\right]x^{n-\frac{1}{4}} - \sum_{n=2}^{\infty} a_{n-2}x^{n-\frac{1}{4}} \\
&= \left(-\frac{3}{8}a_0 + 7a_1\right)x^{\frac{3}{4}} + \sum_{n=2}^{\infty}\left[2(n-\frac{1}{4})(n-\frac{5}{4})a_{n-1} + 4n(n+\frac{3}{4})a_n - a_{n-2}\right]x^{n-\frac{1}{4}}.
\end{aligned}$$

Thus

$$\begin{aligned}
a_1 &= \frac{3}{56}a_0 \\
a_n &= \frac{a_{n-2} - 2(n-\frac{1}{4})(n-\frac{5}{4})a_{n-1}}{4n(n+\frac{3}{4})} \quad \text{for } n \geq 2.
\end{aligned}$$

6. (17 points) Solve the following ordinary differential equation.

$$x^2y'' + xy' + 4y = \sin(\ln x)$$

If $x = e^z$ then the ordinary differential equation can be written as

$$\frac{d^2y}{dz^2} + 4y = \sin z.$$

The complementary solution is $y_c(z) = c_1 \cos 2z + c_2 \sin 2z$ and we will use the method of undetermined coefficients to find a particular solution. If $Y(z) = A \sin z$ then

$$-A \sin z + 4A \sin z = \sin z$$

implies $A = 1/3$. Therefore $y(z) = c_1 \cos 2z + c_2 \sin 2z + \frac{1}{3} \sin z$ and thus

$$y(x) = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x) + \frac{1}{3} \sin(\ln x).$$