Nonhomogeneous Equations; Method of Undetermined Coefficients

MATH 365 Ordinary Differential Equations

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Background

Recently we have concentrated on finding solutions to second order linear, homogeneous ODEs of the form

\[ L[y] = y'' + p(t)y' + q(t)y = 0. \]

Today we return to the case of **nonhomogeneous** second order linear ODEs of the form

\[ L[y] = y'' + p(t)y' + q(t)y = g(t), \]

where \( p \), \( q \), and \( g \) are continuous functions on an open interval \( I \).
Nonhomogeneous vs. Homogeneous Solutions

Theorem

If $Y_1$ and $Y_2$ are two solutions of a nonhomogeneous second order linear ODE, then their difference $Y_1 - Y_2$ is a solution to the corresponding homogeneous second order linear ODE. If, in addition, $y_1$ and $y_2$ are a fundamental set of solutions to the homogeneous second order linear ODE, then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

where $c_1$ and $c_2$ are constants.
Proof

▶ Suppose \( Y_1(t) \) and \( Y_2(t) \) both solve the ODE:

\[
y'' + p(t) y' + q(t) y = g(t).
\]

Let \( \{ y_1(t), y_2(t) \} \) be a fundamental set of solutions to the second order homogeneous linear equation. Since \( y(t) \) is a solution to the homogeneous linear equation, then

\[ y(t) = c_1 y_1(t) + c_2 y_2(t) \]

for some constants \( c_1 \) and \( c_2 \).
Proof

- Suppose \( Y_1(t) \) and \( Y_2(t) \) both solve the ODE:

\[
y'' + p(t) y' + q(t) y = g(t).
\]

- Define \( y(t) = Y_1(t) - Y_2(t) \), then

\[
y'' + p(t) y' + q(t) y
\quad = \quad Y_1''(t) - Y_2''(t) + p(t)(Y_1'(t) - Y_2'(t)) + q(t)(Y_1(t) - Y_2(t))
\quad = \quad Y_1'' + p(t) Y_1' + q(t) Y_1 - Y_2'' - p(t) Y_2' - q(t) Y_2
\quad = \quad g(t) - g(t)
\quad = \quad 0
\]
Proof

▶ Suppose \( Y_1(t) \) and \( Y_2(t) \) both solve the ODE:

\[
y'' + p(t) y' + q(t) y = g(t).
\]

▶ Define \( y(t) = Y_1(t) - Y_2(t) \), then

\[
y'' + p(t) y' + q(t) y
\]
\[
= Y_1''(t) - Y_2''(t) + p(t)(Y_1'(t) - Y_2'(t)) + q(t)(Y_1(t) - Y_2(t))
\]
\[
= Y_1'' + p(t) Y_1' + q(t) Y_1 - Y_2'' - p(t) Y_2' - q(t) Y_2
\]
\[
= g(t) - g(t)
\]
\[
= 0
\]

▶ Let \( \{y_1(t), y_2(t)\} \) be a fundamental set of solutions to the second order homogeneous linear equation. Since \( y(t) \) is a solution to the homogeneous linear equation, then

\[y(t) = c_1 y_1(t) + c_2 y_2(t)\] for some constants \( c_1 \) and \( c_2 \).
The general solution of a nonhomogeneous second order linear ODE can be written as

\[ y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \]

where \( y_1 \) and \( y_2 \) are a fundamental set of solutions to the corresponding homogeneous second order linear ODE, \( c_1 \) and \( c_2 \) are arbitrary constants and \( Y \) is some specific solution to the nonhomogeneous second order linear ODE.
Finding the General Solution

Steps:
1. Find the general solution $c_1 y_1(t) + c_2 y_2(t)$ of the corresponding homogeneous second order linear ODE. This is sometimes called the complementary solution and is denoted $y_c(t)$.

2. Find a single solution $Y(t)$ to the nonhomogeneous second order linear ODE. This is sometimes called the particular solution.

3. Add the complementary and particular solutions.
Method of Undetermined Coefficients

The method of undetermined coefficients can be used to find the particular solution when \( g(t) \) contains

- polynomials,
- sines and cosines (but not the other trigonometric functions),
- exponential functions,

or sums and products of these types of functions.
Example

Find the general solution to the nonhomogeneous second order linear ODE:

\[ y'' + 4y' + 9y = t^2. \]
Solution (1 of 2)

The complementary solution is

\[ y_c(t) = e^{-2t}(c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)). \]
The complementary solution is

\[ y_c(t) = e^{-2t}(c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)). \]

Assume the particular solution has the form

\[ Y(t) = At^2 + Bt + C \]
\[ Y'(t) = 2At + B \]
\[ Y''(t) = 2A \]

and substitute into the nonhomogeneous ODE.
Solution (2 of 2)

\[ t^2 = y'' + 4y' + 9y \]
\[ = 2A + 4(2At + B) + 9(At^2 + Bt + C) \]
\[ = 9At^2 + (8A + 9B)t + (2A + 4B + 9C) \]

Equating coefficients of powers of \( t \) on both sides of the equation, we see that

\[ A = \frac{1}{9}, \quad B = -\frac{8}{81}, \quad C = \frac{14}{729} \]

and thus the general solution to the nonhomogeneous ODE is

\[ y(t) = e^{-2t}(c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)) + \frac{1}{9}t^2 - \frac{8}{81}t + \frac{14}{729} \]
Solution (2 of 2)

\[ t^2 = y'' + 4y' + 9y \]
\[ = 2A + 4(2At + B) + 9(At^2 + Bt + C) \]
\[ = 9At^2 + (8A + 9B)t + (2A + 4B + 9C) \]

Equating coefficients of powers of \( t \) on both sides of the equation, we see that

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Solution (2 of 2)

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\[ A = \frac{1}{9} \]
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and thus the general solution to the nonhomogeneous ODE is

\[ y(t) = e^{-2t}(c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)) + \frac{1}{9} t^2 - \frac{8}{81} t + \frac{14}{729}. \]
Example

Find the general solution to the nonhomogeneous second order linear ODE:

\[ y'' + 4y' + 4y = 6 \sin 3t. \]
The complementary solution is

$$y_c(t) = e^{-2t}(c_1 + c_2 t).$$
The complementary solution is

\[ y_c(t) = e^{-2t}(c_1 + c_2 t). \]

Assume the particular solution has the form

\[
\begin{align*}
Y(t) &= A \cos 3t + B \sin 3t \\
Y'(t) &= -3A \sin 3t + 3B \cos 3t \\
Y''(t) &= -9A \cos 3t - 9B \sin 3t
\end{align*}
\]

and substitute into the nonhomogeneous ODE.
Solution (2 of 2)

\[ 6 \sin 3t = y'' + 4y' + 4y \]
\[ = -9A \cos 3t - 9B \sin 3t + 4(-3A \sin 3t + 3B \cos 3t) \]
\[ + 4(A \cos 3t + B \sin 3t) \]
\[ = (-5A + 12B) \cos 3t + (-12A - 5B) \sin 3t \]
Solution (2 of 2)

\[ 6 \sin 3t = y'' + 4y' + 4y \]
\[ = -9A \cos 3t - 9B \sin 3t + 4(-3A \sin 3t + 3B \cos 3t) \]
\[ + 4(A \cos 3t + B \sin 3t) \]
\[ = (-5A + 12B) \cos 3t + (-12A - 5B) \sin 3t \]

Equating coefficients of the trigonometric functions on both sides of the equation, we have the system of equations:

\[ 0 = -5A + 12B \]
\[ 6 = -12A - 5B. \]
Solution (2 of 2)

\[ 6 \sin 3t = y'' + 4y' + 4y \]
\[ = -9A \cos 3t - 9B \sin 3t + 4(-3A \sin 3t + 3B \cos 3t) \]
\[ + 4(A \cos 3t + B \sin 3t) \]
\[ = (-5A + 12B) \cos 3t + (-12A - 5B) \sin 3t \]

Equating coefficients of the trigonometric functions on both sides of the equation, we have the system of equations:

\[ 0 = -5A + 12B \]
\[ 6 = -12A - 5B. \]

Solving this system yields \( A = -72/169 \) and \( B = -30/169 \) and consequently the general solution to the nonhomogeneous equation is

\[ y(t) = e^{-2t}(c_1 + c_2 t) - \frac{72}{169} \cos 3t - \frac{30}{169} \sin 3t. \]
Example

Find the general solution to the nonhomogeneous second order linear ODE:

\[ y'' - 4y = 4e^{2t}. \]
The complementary solution is

\[ y_c(t) = c_1 e^{-2t} + c_2 e^{2t}. \]
The complementary solution is

\[ y_c(t) = c_1 e^{-2t} + c_2 e^{2t}. \]

The nonhomogeneous term of the ODE is \( g(t) = 4e^{2t} \) which is one of the complementary solutions. Thus we will assume the particular solution has the form

\[ Y(t) = Ate^{2t} \]
\[ Y'(t) = Ae^{2t}(1 + 2t) \]
\[ Y''(t) = Ae^{2t}(4 + 4t) \]

and substitute into the nonhomogeneous ODE.
Solution (2 of 2)

\[ 4e^{2t} = y'' - 4y \]
\[ = Ae^{2t}(4 + 4t) - 4Ate^{2t} \]
\[ 4 = 4A \]
\[ A = 1 \]

Consequently the general solution to the nonhomogeneous equation is
\[ y(t) = c_1 e^{-2t} + c_2 e^{2t} + te^{2t}. \]
4e^{2t} = y'' - 4y
= Ae^{2t}(4 + 4t) - 4Ate^{2t}
4 = 4A
A = 1

Consequently the general solution to the nonhomogeneous equation is

\[ y(t) = c_1 e^{-2t} + c_2 e^{2t} + te^{2t}. \]
Example

Find the solution to the nonhomogeneous second order linear IVP:

\[ y'' + 16y = 5 \sin t \]
\[ y(0) = 0 \]
\[ y'(0) = 0 \]
The complementary solution is

\[ y_c(t) = c_1 \cos 4t + c_2 \sin 4t. \]
The complementary solution is

\[ y_c(t) = c_1 \cos 4t + c_2 \sin 4t. \]

We will assume the particular solution has the form

\[
\begin{align*}
Y(t) & = A \cos t + B \sin t \\
Y'(t) & = -A \sin t + B \cos t \\
Y''(t) & = -A \cos t - B \sin t
\end{align*}
\]

and substitute into the nonhomogeneous ODE.
Solution (2 of 3)

\[ 5 \sin t = y'' + 16y \]
\[ = -A \cos t - B \sin t + 16A \cos t + 16B \sin t \]
\[ = 15A \cos t + 15B \sin t \]

Equating coefficients of the trigonometric functions on both sides of the equation reveals \( A = 0 \) and \( B = \frac{1}{3} \). Consequently, the general solution to the nonhomogeneous equation is 
\[ y(t) = c_1 \cos 4t + c_2 \sin 4t + \frac{1}{3} \sin t. \]

We must use the initial conditions to determine \( c_1 \) and \( c_2 \).
5 \sin t = y'' + 16y \\
= -A \cos t - B \sin t + 16A \cos t + 16B \sin t \\
= 15A \cos t + 15B \sin t \\

Equating coefficients of the trigonometric functions on both sides of the equation reveals \( A = 0 \) and \( B = 1/3 \). Consequently the general solution to the nonhomogeneous equation is 

\[ y(t) = c_1 \cos 4t + c_2 \sin 4t + \frac{1}{3} \sin t. \]
5 \sin t = y'' + 16y
= -A \cos t - B \sin t + 16A \cos t + 16B \sin t
= 15A \cos t + 15B \sin t

Equating coefficients of the trigonometric functions on both sides of the equation reveals \( A = 0 \) and \( B = 1/3 \). Consequently the general solution to the nonhomogeneous equation is

\[ y(t) = c_1 \cos 4t + c_2 \sin 4t + \frac{1}{3} \sin t. \]

We must use the initial conditions to determine \( c_1 \) and \( c_2 \).
Solution (3 of 3)

\[ y(t) = c_1 \cos 4t + c_2 \sin 4t + \frac{1}{3} \sin t \]

\[ y'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t + \frac{1}{3} \cos t \]
Solution (3 of 3)

\[ y(t) = c_1 \cos 4t + c_2 \sin 4t + \frac{1}{3} \sin t \]
\[ y'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t + \frac{1}{3} \cos t \]

Using the given initial conditions we have

\[ 0 = y(0) = c_1 \]
\[ 0 = y'(0) = 4c_2 + \frac{1}{3} \]
Solution (3 of 3)

\[
y(t) = c_1 \cos 4t + c_2 \sin 4t + \frac{1}{3} \sin t
\]

\[
y'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t + \frac{1}{3} \cos t
\]

Using the given initial conditions we have

\[
0 = y(0) = c_1
\]

\[
0 = y'(0) = 4c_2 + \frac{1}{3}
\]

and thus the solution to the IVP is

\[
y(t) = -\frac{1}{12} \sin 4t + \frac{1}{3} \sin t.
\]
Example

Find the general solution to the nonhomogeneous second order linear ODE:

\[ y'' + 4y' + 5y = e^{-t} + 15t. \]
The complementary solution is

\[ y_c(t) = e^{-2t}(c_1 \cos t + c_2 \sin t). \]
The complementary solution is

$$y_c(t) = e^{-2t}(c_1 \cos t + c_2 \sin t).$$

We will assume the particular solution has the form

\[
\begin{align*}
Y(t) &= Ae^{-t} + Bt + C \\
Y'(t) &= -Ae^{-t} + B \\
Y''(t) &= Ae^{-t}
\end{align*}
\]

and substitute into the nonhomogeneous ODE.
Solution (2 of 2)

\[ e^{-t} + 15t = y'' + 4y' + 5y \]
\[ = Ae^{-t} + 4(-Ae^{-t} + B) + 5(Ae^{-t} + Bt + C) \]
\[ = 2Ae^{-t} + 5Bt + (4B + 5C) \]
Solution (2 of 2)

\[ e^{-t} + 15t = y'' + 4y' + 5y \]
\[ = Ae^{-t} + 4(-Ae^{-t} + B) + 5(Ae^{-t} + Bt + C) \]
\[ = 2Ae^{-t} + 5Bt + (4B + 5C) \]

Equating coefficients of the functions on both sides of the equation reveals \( A = \frac{1}{2}, B = 3, \) and \( C = -\frac{12}{5}. \) Consequently the general solution to the nonhomogeneous equation is

\[ y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) + \frac{1}{2}e^{-t} + 3t - \frac{12}{5} \]
Example

Find the general solution to the nonhomogeneous second order linear ODE:

\[ 4y'' + y = t^2 + 2\cos 3t. \]
The complementary solution is

\[ y_c(t) = c_1 \cos\left(\frac{t}{2}\right) + c_2 \sin\left(\frac{t}{2}\right). \]
The complementary solution is

\[ y_c(t) = c_1 \cos(t/2) + c_2 \sin(t/2). \]

We will assume the particular solution has the form

\[
\begin{align*}
Y(t) &= At^2 + Bt + C + D \cos 3t \\
Y'(t) &= 2At + B - 3D \sin 3t \\
Y''(t) &= 2A - 9D \cos 3t 
\end{align*}
\]

and substitute into the nonhomogeneous ODE.
Solution (2 of 2)

\[ t^2 + 2\cos 3t = 4y'' + y \]
\[ = 4(2A - 9D\cos 3t) + At^2 + Bt + C + D\cos 3t \]
\[ = (8A + C) + Bt + At^2 - 35D\cos 3t \]
Solution (2 of 2)

\[ t^2 + 2 \cos 3t = 4y'' + y \]

\[ = 4(2A - 9D \cos 3t) + At^2 + Bt + C + D \cos 3t \]

\[ = (8A + C) + Bt + At^2 - 35D \cos 3t \]

Equating coefficients of the functions on both sides of the equation reveals \( A = 1, \quad B = 0, \quad C = -8, \) and \( D = -2/35. \)

Consequently the general solution to the nonhomogeneous equation is

\[ y(t) = c_1 \cos(t/2) + c_2 \sin(t/2) + t^2 - 8 - \frac{2}{35} \cos 3t. \]
Example

Find the general solution to the nonhomogeneous second order linear ODE:

\[ y'' - 3y' - 4y = -8e^t \sin t. \]
The complementary solution is

\[ y_c(t) = c_1 e^{-t} + c_2 e^{4t}. \]
The complementary solution is

\[ y_c(t) = c_1 e^{-t} + c_2 e^{4t}. \]

We will assume the particular solution has the form

\[
\begin{align*}
Y(t) &= e^t(A \cos t + B \sin t) \\
Y'(t) &= e^t((A + B) \cos t + (-A + B) \sin t) \\
Y''(t) &= 2e^t(B \cos t - A \sin t)
\end{align*}
\]

and substitute into the nonhomogeneous ODE.
Solution (2 of 2)

\[-8e^t \sin t = y'' - 3y' - 4y\]
\[= 2e^t(B \cos t - A \sin t)\]
\[= -3(e^t((A + B) \cos t + (-A + B) \sin t))\]
\[= -4(e^t(A \cos t + B \sin t))\]
\[= -8 \sin t = (-7A - B) \cos t + (A - 7B) \sin t\]
Solution (2 of 2)

\[-8e^t \sin t = y'' - 3y' - 4y \]

\[= 2e^t(B \cos t - A \sin t) - 3(e^t((A + B) \cos t + (-A + B) \sin t)) - 4(e^t(A \cos t + B \sin t))\]

\[-8 \sin t = (-7A - B) \cos t + (A - 7B) \sin t\]

Equating coefficients of the trigonometric functions on both sides of the equation yields the following system of equations.

\[0 = -7A - B\]

\[-8 = A - 7B\]
\[-8e^t \sin t = y'' - 3y' - 4y\]
\[= 2e^t (B \cos t - A \sin t)\]
\[-3(e^t((A + B) \cos t + (-A + B) \sin t))\]
\[-4(e^t(A \cos t + B \sin t))\]
\[-8 \sin t = (-7A - B) \cos t + (A - 7B) \sin t\]

Equating coefficients of the trigonometric functions on both sides of the equation yields the following system of equations.

\[
\begin{align*}
0 &= -7A - B \\
-8 &= A - 7B
\end{align*}
\]

Solving this system produces $A = -4/25$ and $B = 28/25$.

Consequently the general solution to the nonhomogeneous equation is

\[
y(t) = c_1 e^{-t} + c_2 e^{4t} + e^t \left( -\frac{4}{25} \cos 3t + \frac{28}{25} \sin 3t \right).
\]
Example

Find the general solution to the nonhomogeneous second order linear ODE:

\[ y'' - 3y' - 4y = -8e^t \sin t + t \cos t. \]
Find the general solution to the nonhomogeneous second order linear ODE:

\[ y'' - 3y' - 4y = -8e^t \sin t + t \cos t. \]

**Remark:** the homogeneous part of this ODE is the same as in the previous example, thus

\[ y_c(t) = c_1 e^{-t} + c_2 e^{4t}. \]
Solution (1 of 4)

If we solve two nonhomogeneous ODEs:

\[ y'' - 3y' - 4y = -8e^t \sin t \]
\[ y'' - 3y' - 4y = t \cos t \]

and add these particular solutions together they will solve the nonhomogeneous ODE:

\[ y'' - 3y' - 4y = -8e^t \sin t + t \cos t. \]

We solved the first of these nonhomogeneous ODEs in the previous example.

\[ Y_1(t) = e^t \left( -\frac{4}{25} \cos t + \frac{28}{25} \sin t \right) \]
Solution (1 of 4)

If we solve two nonhomogeneous ODEs:

\[ y'' - 3y' - 4y = -8e^t \sin t \]
\[ y'' - 3y' - 4y = t \cos t \]

and add these particular solutions together they will solve the nonhomogeneous ODE:

\[ y'' - 3y' - 4y = -8e^t \sin t + t \cos t. \]

We solved the first of these nonhomogeneous ODEs in the previous example.

\[ Y_1(t) = e^t \left( -\frac{4}{25} \cos t + \frac{28}{25} \sin t \right) \]

We will assume the particular solution to the second nonhomogeneous ODE has the form

\[ Y(t) = (At + B) \cos t + (Ct + D) \sin t \]
\[ Y'(t) = (A + D + Ct) \cos t + (-B + C - At) \sin t \]
\[ Y''(t) = (-B + 2C - At) \cos t + (-2A - D - Ct) \sin t \]
Solution (2 of 4)

\[ t \, \cos t = y'' - 3y' - 4y \]

\[ = (-B + 2C - At) \cos t + (-2A - D - Ct) \sin t \]

\[ - 3((A + D + Ct) \cos t + (-B + C - At) \sin t) \]

\[ - 4((At + B) \cos t + (Ct + D) \sin t) \]

\[ = (- (5A + 3C)t - 3A - 5B + 2C - 3D) \cos t \]

\[ + ((3A - 5C)t - 2A + 3B - 3C - 5D) \sin t \]
Solution (2 of 4)

\[ t \cos t = y'' - 3y' - 4y \]

\[ = (-B + 2C - At) \cos t + (-2A - D - Ct) \sin t \]
\[ - 3((A + D + Ct) \cos t + (-B + C - At) \sin t) \]
\[ - 4((At + B) \cos t + (Ct + D) \sin t) \]
\[ = (-5A + 3C)t - 3A - 5B + 2C - 3D) \cos t \]
\[ + ((3A - 5C)t - 2A + 3B - 3C - 5D) \sin t \]

Equating coefficients of \( t \) on both sides of the equation yields the following system of equations.

\[ 1 = -5A - 3C \]
\[ 0 = 3A - 5C \]
Solution (2 of 4)

\[ t \cos t = y'' - 3y' - 4y \]
\[ = (-B + 2C - At) \cos t + (-2A - D - Ct) \sin t \]
\[ - 3((A + D + Ct) \cos t + (-B + C - At) \sin t) \]
\[ - 4((At + B) \cos t + (Ct + D) \sin t) \]
\[ = (- (5A + 3C)t - 3A - 5B + 2C - 3D) \cos t \]
\[ + ((3A - 5C)t - 2A + 3B - 3C - 5D) \sin t \]

Equating coefficients of \( t \) on both sides of the equation yields the following system of equations.

\[ 1 = -5A - 3C \]
\[ 0 = 3A - 5C \]

Solving this system produces \( A = -5/34 \) and \( C = -3/34 \).
Substituting $A = -5/34$ and $C = -3/34$ into the equation for the undetermined coefficients yields the system of equations:

\[
0 = \frac{9}{34} - 5B - 3D \\
0 = \frac{19}{34} + 3B - 5D
\]

Solving this system yields $B = -3/289$ and $D = 61/578$ and thus the particular solution

\[
Y_2(t) = \left( -\frac{5}{34} t - \frac{3}{289} \right) \cos t + \left( -\frac{3}{34} t + \frac{61}{578} \right) \sin t.
\]
Adding to complementary and both particular solutions produces

\[ y(t) = y_c(t) + Y_1(t) + Y_2(t) \]
\[ = c_1 e^{-t} + c_2 e^{4t} \]
\[ + e^t \left( -\frac{4}{25} \cos t + \frac{28}{25} \sin t \right) \]
\[ + \left( -\frac{5}{34} t - \frac{3}{289} \right) \cos t + \left( -\frac{3}{34} t + \frac{61}{578} \right) \sin t. \]
Example

Find the general solution to the nonhomogeneous second order linear ODE:

\[ y'' - 5y' + 6y = e^{2t}(t + 2)\cos t. \]
The complementary solution is

\[ y_c(t) = c_1 e^{2t} + c_2 e^{3t}. \]
The complementary solution is

\[ y_c(t) = c_1 e^{2t} + c_2 e^{3t}. \]

We will assume the particular solution has the form

\[
Y(t) = e^{2t}(At + B) \cos t + e^{2t}(Ct + D) \sin t \\
Y'(t) = e^{2t}((2A + C)t + A + 2B + D) \cos t \\
\quad + e^{2t}((-A + 2C)t - B + C + 2D) \sin t \\
Y''(t) = e^{2t}((3A + 4C)t + 4A + 3B + 2C + 4D) \cos t \\
\quad + e^{2t}((-4A + 3C)t - 2A - 4B + 4C + 3D) \sin t
\]

and substitute into the nonhomogeneous ODE.
Solution (2 of 3)

\[ e^{2t}(t + 2) \cos t = y'' - 5y' + 6y \]

\[ = e^{2t}((3A + 4C)t + 4A + 3B + 2C + 4D) \cos t \]
\[ + e^{2t}((-4A + 3C)t - 2A - 4B + 4C + 3D) \sin t \]
\[ - 5(e^{2t}((2A + C)t + A + 2B + D) \cos t) \]
\[ - 5(e^{2t}((-A + 2C)t - B + C + 2D) \sin t) \]
\[ + 6(e^{2t}(At + B) \cos t + e^{2t}(Ct + D) \sin t) \]

\[ (t + 2) \cos t = ((-A - C)t - A - B + 2C - D) \cos t \]
\[ + ((A - C)t - 2A + B - C - D) \sin t \]
Solution (2 of 3)

\[ e^{2t}(t + 2) \cos t = y'' - 5y' + 6y \]
\[ = e^{2t}((3A + 4C)t + 4A + 3B + 2C + 4D) \cos t \]
\[ + e^{2t}((−4A + 3C)t − 2A − 4B + 4C + 3D) \sin t \]
\[ − 5(e^{2t}((2A + C)t + A + 2B + D) \cos t) \]
\[ − 5(e^{2t}((−A + 2C)t − B + C + 2D) \sin t) \]
\[ + 6(e^{2t}(At + B) \cos t + e^{2t}(Ct + D) \sin t) \]

\[ (t + 2) \cos t = ((−A − C)t − A − B + 2C − D) \cos t \]
\[ + ((A − C)t − 2A + B − C − D) \sin t \]

Equating coefficients of \( t \) on both sides of the equation yields the following system of equations.

\[
\begin{align*}
1 & = −A − C \\
0 & = A − C
\end{align*}
\]
Solution (2 of 3)

\[ e^{2t}(t + 2) \cos t = y'' - 5y' + 6y \]

\[ = e^{2t}((3A + 4C)t + 4A + 3B + 2C + 4D) \cos t \]
\[ + e^{2t}((-4A + 3C)t - 2A - 4B + 4C + 3D) \sin t \]
\[ - 5(e^{2t}((2A + C)t + A + 2B + D) \cos t) \]
\[ - 5(e^{2t}((-A + 2C)t - B + C + 2D) \sin t) \]
\[ + 6(e^{2t}(At + B) \cos t + e^{2t}(Ct + D) \sin t) \]

\[ (t + 2) \cos t = ((-A - C)t - A - B + 2C - D) \cos t \]
\[ + ((A - C)t - 2A + B - C - D) \sin t \]

Equating coefficients of \( t \) on both sides of the equation yields the following system of equations.

\[ 1 = -A - C \]
\[ 0 = A - C \]

Solving this system produces \( A = -1/2 \) and \( C = -1/2 \).
Solution (3 of 3)

Substituting $A = -1/2$ and $C = -1/2$ into the equation for the undetermined coefficients yields the system of equations:

\[
\begin{align*}
2 & = -\frac{1}{2} - B - D \\
0 & = \frac{3}{2} + B - D
\end{align*}
\]

Solving this system yields $B = -2$ and $D = -1/2$ and thus the particular solution

\[
Y(t) = e^{2t} \left( -\frac{1}{2} t - 2 \right) \cos t + e^{2t} \left( -\frac{1}{2} t - \frac{1}{2} \right) \sin t
\]
Solution (3 of 3)

Substituting $A = -1/2$ and $C = -1/2$ into the equation for the undetermined coefficients yields the system of equations:

$$2 = -\frac{1}{2} - B - D$$
$$0 = \frac{3}{2} + B - D$$

Solving this system yields $B = -2$ and $D = -1/2$ and thus the particular solution

$$Y(t) = e^{2t} \left(-\frac{1}{2}t - 2\right) \cos t + e^{2t} \left(-\frac{1}{2}t - \frac{1}{2}\right) \sin t$$

Finally, the general solution to the nonhomogeneous equation is

$$y(t) = c_1 e^{2t} + c_2 e^{3t} - e^{2t} \left(\frac{1}{2}t + 2\right) \cos t - e^{2t} \left(\frac{1}{2}t + \frac{1}{2}\right) \sin t.$$
Summary

\[ ay'' + by' + cy = g(t) \]

<table>
<thead>
<tr>
<th>( g(t) )</th>
<th>( Y(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_n(t) = a_0 + a_1 t + \cdots + a_n t^n )</td>
<td>( t^k(A_0 + A_1 t + \cdots + A_n t^n) )</td>
</tr>
<tr>
<td>( P_n(t)e^{at} )</td>
<td>( t^k(A_0 + A_1 t + \cdots + A_n t^n)e^{at} )</td>
</tr>
</tbody>
</table>
| \( P_n(t)e^{at} \left\{ \begin{array}{l}
\sin bt \\
\cos bt
\end{array} \right. \) | \( t^k(A_0 + A_1 t + \cdots + A_n t^n)e^{at} \cos bt + t^k(B_0 + B_1 t + \cdots + B_n t^n)e^{at} \sin bt \) |
Homework

- Read Section 3.5
- Exercises: 1–18