Series Solutions Near a Regular Singular Point

MATH 365 Ordinary Differential Equations

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Background

We will find a power series solution to the equation:

\[ P(t)y'' + Q(t)y' + R(t)y = 0. \]

We will assume that \( t_0 \) is a **regular singular point**. This implies:

1. \( P(t_0) = 0, \)
2. \( \lim_{t \to t_0} \frac{(t - t_0)Q(t)}{P(t)} \) exists,
3. \( \lim_{t \to t_0} \frac{(t - t_0)^2R(t)}{P(t)} \) exists.
Simplification

If $t_0 \neq 0$ then we can make the change of variable $x = t - t_0$ and the ODE:

$$P(x + t_0)y'' + Q(x + t_0)y' + R(x + t_0)y = 0.$$  

has a regular singular point at $x = 0$.

From now on we will work with the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

having a regular singular point at $x = 0$. 
Assumptions (1 of 2)

Since the ODE has a regular singular point at $x = 0$ we can define

$$x \frac{Q(x)}{P(x)} = xp(x) \quad \text{and} \quad x^2 \frac{R(x)}{P(x)} = x^2 q(x)$$

which are analytic at $x = 0$ and

$$\lim_{x \to 0} xQ(x) = \lim_{x \to 0} xp(x) = p_0$$

$$\lim_{x \to 0} x^2 R(x) = \lim_{x \to 0} x^2 q(x) = q_0.$$
Assumptions (2 of 2)

Furthermore since \( xp(x) \) and \( x^2q(x) \) are analytic,

\[
xp(x) = \sum_{n=0}^{\infty} p_n x^n
\]

\[
x^2q(x) = \sum_{n=0}^{\infty} q_n x^n
\]

for all \(-\rho < x < \rho\) with \(\rho > 0\).
Re-writing the ODE

The second order linear homogeneous ODE can be written as

\[ 0 = P(x)y'' + Q(x)y' + R(x)y \]
\[ = y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y \]
\[ = x^2 y'' + x^2 \frac{Q(x)}{P(x)}y' + x^2 \frac{R(x)}{P(x)}y \]
\[ = x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y \]
\[ = x^2 y'' + x [p_0 + p_1 x + \cdots + p_n x^n + \cdots] y' \]
\[ + [q_0 + q_1 x + \cdots + q_n x^n + \cdots] y. \]
If $p_n = 0$ and $q_n = 0$ for $n \geq 1$ then

$$0 = x^2 y'' + x \left[ p_0 + p_1 x + \cdots + p_n x^n + \cdots \right] y'$$

$$+ \left[ q_0 + q_1 x + \cdots + q_n x^n + \cdots \right] y$$

$$= x^2 y'' + p_0 xy' + q_0 y$$

which is Euler’s equation.
General Case

When \( p_n \neq 0 \) and/or \( q_n \neq 0 \) for some \( n > 0 \) then we will assume the solution to

\[
x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0
\]

has the form

\[
y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n},
\]

an Euler solution multiplied by a power series.
Solution Procedure

Assuming $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ we must determine:

1. the values of $r$,
2. a recurrence relation for $a_n$,
3. the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$. 
Example (1 of 8)

Consider the following ODE for which $x = 0$ is a regular singular point.

$$4xy'' + 2y' + y = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$ is a solution, determine the values of $r$ and $a_n$ for $n \geq 0$. 
Example (1 of 8)

Consider the following ODE for which \( x = 0 \) is a regular singular point.

\[
4xy'' + 2y' + y = 0
\]

Assuming \( y(x) = \sum_{n=0}^{\infty} a_n x^{r+n} \) is a solution, determine the values of \( r \) and \( a_n \) for \( n \geq 0 \).

\[
y'(x) = \sum_{n=0}^{\infty} (r + n) a_n x^{r+n-1}
\]

\[
y''(x) = \sum_{n=0}^{\infty} (r + n)(r + n - 1) a_n x^{r+n-2}
\]
Example (2 of 8)

\[ 0 = 4xy'' + 2y' + y \]

\[ = 4x \sum_{n=0}^{\infty} (r + n)(r + n - 1)a_n x^{r+n-2} + 2 \sum_{n=0}^{\infty} (r + n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \]

\[ = \sum_{n=0}^{\infty} 4(r + n)(r + n - 1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} 2(r + n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \]

\[ = \sum_{n=0}^{\infty} [4(r + n)(r + n - 1) + 2(r + n)] a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \]
$$0 \ = \ \sum_{n=0}^{\infty} \left[ 4(r + n)(r + n - 1) + 2(r + n) \right] a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n}$$

$$\ = \ \sum_{n=0}^{\infty} 2a_n(r + n)(2r + 2n - 1)x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n}$$

$$\ = \ \sum_{n=0}^{\infty} 2a_n(r + n)(2r + 2n - 1)x^{r+n-1} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n-1}$$
Example (4 of 8)

\[
0 = \sum_{n=0}^{\infty} 2a_n(r + n)(2r + 2n - 1)x^{r+n-1} + \sum_{n=1}^{\infty} a_{n-1}x^{r+n-1}
\]

\[
= 2a_0r(2r - 1)x^{r-1} + \sum_{n=1}^{\infty} 2a_n(r + n)(2r + 2n - 1)x^{r+n-1}
\]

\[
+ \sum_{n=1}^{\infty} a_{n-1}x^{r+n-1}
\]

\[
= 2a_0r(2r - 1)x^{r-1} + \sum_{n=1}^{\infty} [2a_n(r + n)(2r + 2n - 1) + a_{n-1}]x^{r+n-1}
\]
Example (5 of 8)

\[ 0 = 2a_0 r(2r - 1)x^{r-1} + \sum_{n=1}^{\infty} [2a_n(r + n)(2r + 2n - 1) + a_{n-1}] x^{r+n-1} \]

This implies

\[ 0 = r(2r - 1) \quad \text{(the indicial equation)} \quad \text{and} \]
\[ 0 = 2a_n(r + n)(2r + 2n - 1) + a_{n-1} \]

Thus we see that \( r = 0 \) or \( r = \frac{1}{2} \) and the recurrence relation is

\[ a_n = -\frac{a_{n-1}}{(2r + 2n)(2r + 2n - 1)}, \quad \text{for} \ n \geq 1. \]
Example, Case $r = 0$ (6 of 8)

The recurrence relation becomes $a_n = -\frac{a_{n-1}}{2n(2n-1)}$.

\[
\begin{align*}
a_1 &= -\frac{a_0}{(2)(1)} = -\frac{a_0}{2!} \\
a_2 &= -\frac{a_1}{(4)(3)} = \frac{a_0}{4!} \\
a_3 &= -\frac{a_2}{(6)(5)} = -\frac{a_0}{6!} \\
\vdots \\
a_n &= (-1)^n a_0 \\
&= \frac{a_0}{(2n)!}
\end{align*}
\]
Example, Case \( r = 0 \) (6 of 8)

The recurrence relation becomes
\[
a_n = -\frac{a_{n-1}}{2n(2n-1)}.
\]

Thus
\[
y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n)!} x^{n+0} = a_0 \cos \sqrt{x}.
\]
Example, Case $r = 1/2$ (7 of 8)

The recurrence relation becomes $a_n = -\frac{a_{n-1}}{(2n+1)2n}$.

\[
a_1 = -\frac{a_0}{(3)(2)} = -\frac{a_0}{3!}
\]
\[
a_2 = -\frac{a_1}{(5)(4)} = \frac{a_0}{5!}
\]
\[
a_3 = -\frac{a_2}{(7)(6)} = -\frac{a_0}{7!}
\]
\[
\vdots
\]
\[
a_n = \frac{(-1)^n a_0}{(2n+1)!}
\]
The recurrence relation becomes \( a_n = -\frac{a_{n-1}}{(2n+1)2n} \).

\[
\begin{align*}
a_1 &= -\frac{a_0}{(3)(2)} = -\frac{a_0}{3!} \\
a_2 &= -\frac{a_1}{(5)(4)} = \frac{a_0}{5!} \\
a_3 &= -\frac{a_2}{(7)(6)} = -\frac{a_0}{7!} \\
&\vdots \\
a_n &= \frac{(-1)^n a_0}{(2n+1)!}
\end{align*}
\]

Thus \( y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n+1)!} x^{n+1/2} = a_0 \sin \sqrt{x} \).
We should verify that the general solution to

\[ 4xy'' + 2y' + y = 0 \]

is

\[ y(x) = c_1 \cos \sqrt{x} + c_2 \sin \sqrt{x}. \]
Remarks

- This technique just outlined will succeed provided $r_1 \neq r_2$ and $r_1 - r_2 \neq n \in \mathbb{Z}$.
- If $r_1 = r_2$ or $r_1 - r_2 = n \in \mathbb{Z}$ then we can always find the solution corresponding to the larger of the two roots $r_1$ or $r_2$.
- The second (linearly independent) solution will have a more complicated form involving $\ln x$. 
Homework

- Read Section 5.5
- Exercises: 3, 6, 11–13
General Case: Method of Frobenius

Given \( x^2 y'' + x [xp(x)] y' + \left[ x^2 q(x) \right] y = 0 \) where \( x = 0 \) is a regular singular point and

\[
xp(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n
\]

are analytic at \( x = 0 \), we will seek a solution to the ODE of the form

\[
y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}
\]

where \( a_0 \neq 0 \).
Substitute into the ODE

\[ 0 = x^2 \sum_{n=0}^{\infty} (r + n)(r + n - 1)a_n x^{r+n-2} \]
\[ + x \left[ \sum_{n=0}^{\infty} p_n x^n \right] \sum_{n=0}^{\infty} (r + n)a_n x^{r+n-1} + \left[ \sum_{n=0}^{\infty} q_n x^n \right] \sum_{n=0}^{\infty} a_n x^{r+n} \]
\[ = \sum_{n=0}^{\infty} (r + n)(r + n - 1)a_n x^{r+n} \]
\[ + \left[ \sum_{n=0}^{\infty} p_n x^n \right] \sum_{n=0}^{\infty} (r + n)a_n x^{r+n} + \left[ \sum_{n=0}^{\infty} q_n x^n \right] \sum_{n=0}^{\infty} a_n x^{r+n} \]
Collect Like Powers of $x$

\[ 0 = a_0 r (r - 1) x^r + a_1 (r + 1) r x^{r+1} + \cdots \]
\[ + (p_0 + p_1 x + \cdots) (a_0 r x^r + a_1 (r + 1) r x^{r+1} + \cdots) \]
\[ + (q_0 + q_1 x + \cdots) (a_0 x^r + a_1 x^{r+1} + \cdots) \]
\[ = a_0 [r(r - 1) + p_0 r + q_0] x^r \]
\[ + [a_1 (r + 1) r + p_0 a_1 (r + 1) + p_1 a_0 r + q_0 a_1 + q_1 a_0] x^{r+1} \]
\[ + \cdots \]
\[ = a_0 [r(r - 1) + p_0 r + q_0] x^r \]
\[ + [a_1 ((r + 1) r + p_0 (r + 1) + q_0) + a_0 (p_1 r + q_1)] x^{r+1} \]
\[ + \cdots \]
**Indicial Equation**

If we define $F(r) = r(r - 1) + p_0 r + q_0$ then the ODE can be written as

$$0 = a_0 F(r) x^r + [a_1 F(r + 1) + a_0 (p_1 r + q_1)] x^{r+1} + [a_2 F(r + 2) + a_0 (p_2 r + q_2) + a_1 (p_1 (r + 1) + q_1)] x^{r+2} + \cdots$$

The equation

$$0 = F(r) = r(r - 1) + p_0 r + q_0$$

is called the **indicial equation**. The solutions are called the **exponents of singularity**.
The coefficients of \( x^{r+n} \) for \( n \geq 1 \) determine the recurrence relation:

\[
0 = a_n F(r + n) + \sum_{k=0}^{n-1} a_k \left( p_{n-k}(r + k) + q_{n-k} \right)
\]

\[
a_n = -\sum_{k=0}^{n-1} a_k \left( p_{n-k}(r + k) + q_{n-k} \right) / F(r + n)
\]

provided \( F(r + n) \neq 0 \).
Exponents of Singularity

- By convention we will let the roots of the indicial equation $F(r) = 0$ be $r_1$ and $r_2$. 

- Consequently the recurrence relation where $r = r_1$, $a_n(r_1) = -\sum_{k=0}^{n-1} a_k(p_n-k(r_1+k) + q_n-k)$ $F(r_1+n)$ is defined for all $n \geq 1$.

- One solution to the ODE is then $y_1(x) = x^{r_1}(1+\infty \sum_{n=1} a_n(r_1)x^n)$. 
Exponents of Singularity

- By convention we will let the roots of the indicial equation $F(r) = 0$ be $r_1$ and $r_2$.
- When $r_1$ and $r_2 \in \mathbb{R}$ we will assign subscripts so that $r_1 \geq r_2$. 

\[ a_n(r_1) = -\sum_{k=0}^{\infty} a_k(p_n - k(r_1 + k)) + q_n(r_1 + n) F(r_1 + n) \] 

One solution to the ODE is then 

\[ y_1(x) = x^{r_1}(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n) \]
Exponents of Singularity

By convention we will let the roots of the indicial equation $F(r) = 0$ be $r_1$ and $r_2$.

When $r_1$ and $r_2 \in \mathbb{R}$ we will assign subscripts so that $r_1 \geq r_2$.

Consequently the recurrence relation where $r = r_1$,

$$a_n(r_1) = -\sum_{k=0}^{n-1} a_k (p_{n-k}(r_1 + k) + q_{n-k}) \frac{F(r_1 + n)}{F(r_1 + n)}$$

is defined for all $n \geq 1$. 
Exponents of Singularity

- By convention we will let the roots of the indicial equation $F(r) = 0$ be $r_1$ and $r_2$.
- When $r_1$ and $r_2 \in \mathbb{R}$ we will assign subscripts so that $r_1 \geq r_2$.
- Consequently the recurrence relation where $r = r_1$,

$$a_n(r_1) = -\sum_{k=0}^{n-1} a_k (p_{n-k}(r_1 + k) + q_{n-k})$$

$$F(r_1 + n)$$

is defined for all $n \geq 1$.
- One solution to the ODE is then

$$y_1(x) = x^{r_1} \left(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right).$$
Case: $r_1 - r_2 \not\in \mathbb{N}$

- If $r_1 - r_2 \neq n$ for any $n \in \mathbb{N}$ then $r_1 \neq r_2 + n$ for any $n \in \mathbb{N}$ and consequently $F(r_2 + n) \neq 0$ for any $n \in \mathbb{N}$. 

A second solution to the ODE is then

$$y_2(x) = x^{r_2}(1 + \sum_{n=1}^{\infty} a_n(r_2)x^n).$$
Case: $r_1 - r_2 \notin \mathbb{N}$

- If $r_1 - r_2 \neq n$ for any $n \in \mathbb{N}$ then $r_1 \neq r_2 + n$ for any $n \in \mathbb{N}$ and consequently $F(r_2 + n) \neq 0$ for any $n \in \mathbb{N}$.
- Consequently the recurrence relation where $r = r_2$,

$$a_n(r_2) = -\sum_{k=0}^{n-1} a_k (p_{n-k}(r_2 + k) + q_{n-k})$$

$$\frac{F(r_2 + n)}{F(r_2 + n)}$$

is defined for all $n \geq 1$. 
Case: $r_1 - r_2 \not\in \mathbb{N}$

- If $r_1 - r_2 \neq n$ for any $n \in \mathbb{N}$ then $r_1 \neq r_2 + n$ for any $n \in \mathbb{N}$ and consequently $F(r_2 + n) \neq 0$ for any $n \in \mathbb{N}$.
- Consequently the recurrence relation where $r = r_2$,

$$a_n(r_2) = -\sum_{k=0}^{n-1} a_k (p_{n-k}(r_2 + k) + q_{n-k}) \cdot \frac{F(r_2 + n)}{F(r_2 + n)}$$

is defined for all $n \geq 1$.

- A second solution to the ODE is then

$$y_2(x) = x^{r_2} \left( 1 + \sum_{n=1}^{\infty} a_n(r_2)x^n \right).$$
Example

Find the indicial equation, exponents of singularity, and discuss the nature of solutions to the ODE

\[ x^2 y'' - x(2 + x)y' + (2 + x^2)y = 0 \]

near the regular singular point \( x = 0 \).
Solution

\[ p_0 = \lim_{x \to 0} x \frac{-x(2 + x)}{x^2} = - \lim_{x \to 0} (2 + x) = -2 \]

\[ q_0 = \lim_{x \to 0} x^2 \frac{2 + x^2}{x^2} = \lim_{x \to 0} (2 + x^2) = 2 \]
Solution

\[ p_0 = \lim_{x \to 0} x \frac{-x(2 + x)}{x^2} = - \lim_{x \to 0} (2 + x) = -2 \]

\[ q_0 = \lim_{x \to 0} x^2 \frac{2 + x^2}{x^2} = \lim_{x \to 0} (2 + x^2) = 2 \]

The indicial equation is then

\[ r(r - 1) + (-2)r + 2 = 0 \]
\[ r^2 - 3r + 2 = 0 \]
\[ (r - 2)(r - 1) = 0. \]
Solution

\[ p_0 = \lim_{x \to 0} x \frac{-x(2 + x)}{x^2} = - \lim_{x \to 0} (2 + x) = -2 \]

\[ q_0 = \lim_{x \to 0} x^2 \frac{2 + x^2}{x^2} = \lim_{x \to 0} (2 + x^2) = 2 \]

The indicial equation is then

\[ r(r - 1) + (-2)r + 2 = 0 \]

\[ r^2 - 3r + 2 = 0 \]

\[ (r - 2)(r - 1) = 0. \]

The exponents of singularity are \( r_1 = 2 \) and \( r_2 = 1 \). Consequently we have one solution of the form

\[ y_1(x) = x^2 \left( 1 + \sum_{n=1}^{\infty} a_n x^n \right). \]
Case: \( r_1 = r_2 \) Equal Exponents of Singularity (1 of 4)

- When the exponents of singularity are equal then
  \[ F(r) = (r - r_1)^2. \]
Case: $r_1 = r_2$ Equal Exponents of Singularity (1 of 4)

- When the exponents of singularity are equal then $F(r) = (r - r_1)^2$.
- We have a solution to the ODE of the form

$$y_1(x) = x^r \left( 1 + \sum_{n=1}^{\infty} a_n(r) x^n \right).$$
Case: $r_1 = r_2$ Equal Exponents of Singularity (1 of 4)

- When the exponents of singularity are equal then $F(r) = (r - r_1)^2$.
- We have a solution to the ODE of the form

$$y_1(x) = x^r \left(1 + \sum_{n=1}^{\infty} a_n(r)x^n\right).$$

- Differentiating this solution and substituting into the ODE yields

$$0 = a_0 F(r)x^r$$
$$+ \sum_{n=1}^{\infty} \left[a_n F(r + n) + \sum_{k=0}^{n-1} a_k (p_{n-k}(r + k) + q_{n-k})\right] x^{r+n}$$
$$= a_0 (r - r_1)^2 x^r.$$

when $a_n$ solves the recurrence relation.
Recall: for the ODE \( x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0 \) we can define the linear operator

\[
L[y] = x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y
\]

so that the ODE can be written compactly as \( L[y] = 0 \).
Case: $r_1 = r_2$ Equal Exponents of Singularity (2 of 4)

**Recall:** for the ODE $x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0$ we can define the **linear operator**

$$L[y] = x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y$$

so that the ODE can be written compactly as $L[y] = 0$.

Consider the infinite series solution to the ODE,

$$\phi(r, x) = x^r \left[1 + \sum_{n=1}^{\infty} a_n(r)x^n\right].$$

**Note:** since the coefficients of the series depend on $r$ we denote the solution as $\phi(r, x)$. 
Case: $r_1 = r_2$ Equal Exponents of Singularity (3 of 4)

We have seen that $\phi(r_1, x)$ solve the ODE and now we claim that $\phi_r(r_1, x)$ solves it as well.

\[
L \left[ \frac{\partial \phi}{\partial r} \right] (r_1, x) = \frac{\partial}{\partial r} \left[ \left( a_0 (r - r_1)^2 x^r \right) \right]_{r=r_1} = a_0 \left[ 2(r - r_1)x^r + (r - r_1)^2 (\ln x)x^r \right]_{r=r_1} = 0
\]
Case: \( r_1 = r_2 \) Equal Exponents of Singularity (3 of 4)

We have seen that \( \phi(r_1, x) \) solve the ODE and now we claim that \( \phi_r(r_1, x) \) solves it as well.

\[
L \left[ \frac{\partial \phi}{\partial r} \right] (r_1, x) = \frac{\partial}{\partial r} \left[ (a_0(r - r_1)^2x^r) \right]_{r=r_1} \\
= a_0 \left[ 2(r - r_1)x^r + (r - r_1)^2(\ln x)x^r \right]_{r=r_1} \\
= 0
\]

Thus a second solution to the ODE is \( y_2(x) = \frac{\partial \phi(r, x)}{\partial r} \bigg|_{r=r_1} \).
Case: $r_1 = r_2$ Equal Exponents of Singularity (4 of 4)

$$y_2(x) = \frac{\partial \phi(r, x)}{\partial r} \bigg|_{r=r_1}$$

$$= \frac{\partial}{\partial r} \left[ x^r \left( 1 + \sum_{n=1}^{\infty} a_n(r)x^n \right) \right]_{r=r_1}$$

$$= (\ln x)x^r \left( 1 + \sum_{n=1}^{\infty} a_n(r)x^n \right) + x^r \sum_{n=1}^{\infty} a'_n(r)x^n \bigg|_{r=r_1}$$

$$= (\ln x)y_1(x) + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1)x^n$$
Example (1 of 9)

Find the general solution to the ODE:

$$xy'' + y' + xy = 0$$

near the regular singular point $x = 0$. 
Example (1 of 9)

Find the general solution to the ODE:

\[ xy'' + y' + xy = 0 \]

near the regular singular point \( x = 0 \).

\[
\lim_{{x \to 0}} x \left( \frac{1}{x} \right) = 1 = p_0
\]

\[
\lim_{{x \to 0}} x^2 \left( \frac{x}{x} \right) = 0 = q_0
\]

Thus the indicial equation is \( F(r) = r(r - 1) + r = r^2 = 0 \) and the exponents of singularity are \( r_1 = r_2 = 0 \).
Example (2 of 9)

Assume \( y(x) = \sum_{n=0}^{\infty} a_n x^{r+n} \), differentiate, and substitute into the given ODE.
Example (2 of 9)

Assume \( y(x) = \sum_{n=0}^{\infty} a_n x^{r+n} \), differentiate, and substitute into the given ODE.

\[
0 = x \sum_{n=0}^{\infty} (r + n)(r + n - 1) a_n x^{r+n-2} + \sum_{n=0}^{\infty} (r + n) a_n x^{r+n-1} \\
+ x \sum_{n=0}^{\infty} a_n x^{r+n} \\
= \sum_{n=0}^{\infty} (r + n)(r + n - 1) a_n x^{r+n-1} + \sum_{n=0}^{\infty} (r + n) a_n x^{r+n-1} \\
+ \sum_{n=0}^{\infty} a_n x^{r+n+1} \\
= \sum_{n=0}^{\infty} (r + n)^2 a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n+1}
\]
0 = \sum_{n=0}^{\infty} (r + n)^2 a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n+1}

= \sum_{n=0}^{\infty} (r + n)^2 a_n x^{r+n-1} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n-1}

= a_0 r^2 x^{r-1} + a_1 (r + 1)^2 x^r + \sum_{n=2}^{\infty} (r + n)^2 a_n x^{r+n-1}

+ \sum_{n=2}^{\infty} a_{n-2} x^{r+n-1}

= a_0 r^2 x^{r-1} + a_1 (r + 1)^2 x^r + \sum_{n=2}^{\infty} \left[ (r + n)^2 a_n + a_{n-2} \right] x^{r+n-1}
The exponents of singularity are $r_1 = r_2 = 0$.

The recurrence relation is $a_n(r) = -\frac{a_{n-2}(r)}{(r + n)^2}$.

$a_1 = 0$ which implies $a_{2n+1} = 0$ for all $n \in \mathbb{N}$. 

\[
0 = a_0 r^2 x^{r-1} + a_1 (r + 1)^2 x^r + \sum_{n=2}^{\infty} \left[ (r + n)^2 a_n + a_{n-2} \right] x^{r+n-1}
\]
When $r = 0$, and $a_0$ is arbitrary

\[
\begin{align*}
    a_2 &= -\frac{a_0}{2^2} = -\frac{a_0}{4^1(1!)^2} \\
    a_4 &= -\frac{a_2}{4^2} = \frac{a_0}{4^2(2!)^2} \\
    a_6 &= -\frac{a_4}{6^2} = -\frac{a_0}{4^3(3!)^2} \\
    \vdots \\
    a_{2n} &= \frac{(-1)^n a_0}{4^n(n!)^2}
\end{align*}
\]

thus

\[
y_1(x) = a_0 \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{4^n(n!)^2} \right).
\]
Now find the second solution.

\[ a_n(r) = - \frac{a_{n-2}(r)}{(r + n)^2} \]

\[ a'_n(r) = - \frac{a'_{n-2}(r)(r + n)^2 - a_{n-2}(r)2(r + n)}{(r + n)^4} \]

\[ = - \frac{a'_{n-2}(r)(r + n) - 2a_{n-2}(r)}{(r + n)^3} \]

\[ a'_n(0) = \frac{2a_{n-2}(0) - na'_{n-2}(0)}{n^3} \]
Example (7 of 9)

Since $a_{2n+1}(r) = 0$ for all $n \in \mathbb{N}$ then $a'_{2n+1}(r) = 0$ for all $n \in \mathbb{N}$. Since $a_0$ is an arbitrary constant then $a'_0 = 0$. 
Example (8 of 9)
Recall the recurrence relation for \( n \geq 2 \):
\[
a'_n(0) = \frac{2a_{n-2}(0) - na'_{n-2}(0)}{n^3}
\]

If \( n = 2 \) then
\[
a'_2(0) = \frac{2a_0 - 2a'_0}{2^3}
= \frac{a_0}{4} = (1)\frac{a_0}{4^1(1!)^2}
\]

If \( n = 4 \) then
\[
a'_4(0) = \frac{2a_2 - 4a'_2}{4^3}
= \frac{a_2 - 2a'_2}{4^2(2!)}
= \frac{1}{4^2(2!)} \left(- \frac{a_0}{4} - 2 \left(\frac{a_0}{4}\right)\right)
= - \left(1 + \frac{1}{2}\right) \frac{a_0}{4^2(2!)^2}
\]
Example (9 of 9)

\[
a'_6(0) = \frac{2a_4 - 6a'_4}{6^3} = \frac{a_4 - 3a'_4}{6^2(3)} = \left(1 + \frac{1}{2} + \frac{1}{3}\right) \frac{a_0}{4^3(3!)^2}
\]

\[
\vdots
\]

\[
a'_{2n}(0) = \frac{(-1)^{n+1} \sum_{k=1}^{n} \frac{1}{k}}{4^n(n!)^2}
\]

Thus

\[
y_2(x) = (\ln x)y_1(x) + \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1} \sum_{k=1}^{n} \frac{1}{k}}{4^n(n!)^2}\right) x^{2n}.
\]
Homework

- Read Section 5.5
- Exercises: 13
Considering the second-order linear, homogeneous ODE:

\[ P(x)y'' + Q(x)y' + R(x)y = 0 \]

where \( x_0 = 0 \) is a regular singular point.

This implies \( P(x_0) = 0 \) and

\[
\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} = \lim_{x \to 0} x p(x) = p_0
\]

\[
\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2 q(x) = q_0.
\]
Define the polynomial $F(r) = r(r - 1) + p_0 r + q_0$, then

$$r(r - 1) + p_0 r + q_0 = 0$$

is called the **indicial equation** and the roots $r_1 \geq r_2$ are called the **exponents of singularity**.
Define the polynomial \( F(r) = r(r - 1) + p_0 r + q_0 \), then
\[
r(r - 1) + p_0 r + q_0 = 0
\]
is called the \textbf{indicial equation} and the roots \( r_1 \geq r_2 \) are called the \textbf{exponents of singularity}.

If \( r_1 - r_2 \notin \mathbb{N} \) then we have a fundamental set of solutions of the form
\[
y_1(x) = x^{r_1} \left[ 1 + \sum_{n=1}^{\infty} a_n(r_1)x^n \right]
\]
\[
y_2(x) = x^{r_2} \left[ 1 + \sum_{n=1}^{\infty} a_n(r_2)x^n \right].
\]
If $r_1 = r_2$ then we have a fundamental set of solutions of the form

$$y_1(x) = x^{r_1} \left[ 1 + \sum_{n=1}^{\infty} a_n(r_1)x^n \right]$$

$$y_2(x) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1)x^n.$$
If $r_1 = r_2$ then we have a fundamental set of solutions of the form

$$y_1(x) = x^{r_1} \left[ 1 + \sum_{n=1}^{\infty} a_n(r_1) x^n \right]$$

$$y_2(x) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n.$$ 

Now we may take up the final case when $r_1 - r_2 \in \mathbb{N}$. 

Case: $r_1 - r_2 = N \in \mathbb{N}$

The second solution has the form

$$y_2(x) = a y_1(x) \ln x + x^{r_2} \left[ 1 + \sum_{n=1}^{\infty} c_n(r_2) x^n \right]$$

where

$$a = \lim_{r \to r_2} (r - r_2) a_N(r)$$

and

$$c_n(r_2) = \frac{d}{dr} [(r - r_2) a_n(r)]_{r=r_2}.$$

We can assume $a_0 = 1$ for simplicity.
Find the general solution to the ODE
\[ x \, y'' - y = 0 \]
with regular singular point at \( x = 0 \).
Find the general solution to the ODE

\[ x \, y'' - y = 0 \]

with regular singular point at \( x = 0 \).

\[ \lim_{x \to 0} x \left( \frac{0}{x} \right) = 0 = p_0 \]

\[ \lim_{x \to 0} x^2 \left( \frac{-1}{x} \right) = 0 = q_0 \]

Thus the indicial equation is \( F(r) = r(r - 1) \) and the exponents of singularity are \( r_1 = 1 \) and \( r_2 = 0 \).
Example (2 of 8)

\[ 0 = x \sum_{n=0}^{\infty} (r + n)(r + n - 1)a_n x^{r + n - 2} - \sum_{n=0}^{\infty} a_n x^{r + n} \]

\[ = \sum_{n=0}^{\infty} (r + n)(r + n - 1)a_n x^{r + n - 1} - \sum_{n=0}^{\infty} a_n x^{r + n} \]

\[ = \sum_{n=0}^{\infty} (r + n)(r + n - 1)a_n x^{r + n - 1} - \sum_{n=0}^{\infty} a_{n-1} x^{r + n - 1} \]

\[ = a_0 r(r - 1)x^{r-1} + \sum_{n=1}^{\infty} [(r + n)(r + n - 1)a_n - a_{n-1}] x^{r + n - 1} \]
Example (3 of 8)

Recurrence relation for $n \geq 1$:

\[
\begin{align*}
a_n(r) &= \frac{a_{n-1}(r)}{(r + n)(r + n - 1)} \\
a_n(1) &= \frac{a_{n-1}(1)}{n(n + 1)}
\end{align*}
\]

If $a_0 = 1$ then

\[
a_n(1) = \frac{1}{n!(n + 1)!}
\]

and

\[
y_1(x) = x \left[ 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!(n + 1)!} \right].
\]
Example (4 of 8)

According to the formula of Frobenius

\[ y_2(x) = ay_1(x) \ln x + x^{r_2} \left[ 1 + \sum_{n=1}^{\infty} c_n(r_2)x^n \right]. \]

\[
\begin{align*}
a &= \lim_{r \to r_2} \frac{1}{a_N(r)} (r - r_2) \\
&= \lim_{r \to 0} ra_1(r) \\
&= \lim_{r \to 0} r \frac{a_0}{r(r + 1)} \\
&= \lim_{r \to 0} \frac{1}{r + 1} \\
&= 1
\end{align*}
\]
Example (5 of 8)

c_1(r_2) = \frac{d}{dr} [(r - r_2)a_1(r)]_{r=r_2}

c_1(0) = \frac{d}{dr} \left[ \frac{r a_0}{r(r + 1)} \right]_{r=0}

= \frac{d}{dr} \left[ \frac{a_0}{r + 1} \right]_{r=0}

= \frac{d}{dr} \left[ \frac{1}{r + 1} \right]_{r=0}

= -1
Example (6 of 8)

\[
c_2(r_2) = \left. \frac{d}{dr} \left[ (r - r_2)a_2(r) \right] \right|_{r=r_2}
\]

\[
c_2(0) = \left. \frac{d}{dr} \left[ ra_2(r) \right] \right|_{r=0}
\]

\[
= \left. \frac{d}{dr} \left[ \frac{ra_1(r)}{(r + 1)(r + 2)} \right] \right|_{r=0}
\]

\[
= \left. \frac{d}{dr} \left[ \frac{ra_0}{r(r + 1)^2(r + 2)} \right] \right|_{r=0}
\]

\[
= \left. \frac{d}{dr} \left[ \frac{1}{(r + 1)^2(r + 2)} \right] \right|_{r=0}
\]

\[
= -\frac{5}{4}
\]
Example (7 of 8)

\[ c_3(r_2) = \frac{d}{dr} [(r - r_2)a_3(r)]_{r=r_2} \]

\[ c_3(0) = \frac{d}{dr} \left[ \frac{1}{(r + 1)^2(r + 2)^2(r + 3)} \right]_{r=0} = -\frac{5}{18} \]
So the second solution has the form

\[ y_2(x) = y_1(x) \ln x + 1 - x - \frac{5}{4} x^2 - \frac{5}{18} x^3 + \cdots. \]
Homework

- Read Section 5.6
- Exercise: 1, 5, 9, 15, 19