1 Series Solutions Near Regular Singular Points

All of the work here will be directed toward finding series solutions of a second order linear homogeneous ordinary differential equation:

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{1}$$

We will assume that x_0 is a **regular singular point** for this equation. The definition of regular singular point implies the following three things:

1. $P(x_0) = 0$,
2. $\displaystyle\lim_{x \to x_0} \frac{(x - x_0)Q(x)}{P(x)}$ exists,
3. $\displaystyle\lim_{x \to x_0} \frac{(x - x_0)^2 R(x)}{P(x)}$ exists.

To keep things as simple as possible in the following discussion we will let $x_0 = 0$ be a regular singular point and

$$\lim_{x \to 0} \frac{xQ(x)}{P(x)} = p_0 \qquad \text{and} \qquad \lim_{x \to 0} \frac{x^2 R(x)}{P(x)} = q_0.$$

Thus in an interval containing $x_0 = 0$ we can divide both sides of equation (1) by $P(x)$ and multiply by x^2 to rewrite equation (1) in the following way.

$$x^2 y'' + x \left[\frac{xQ(x)}{P(x)}\right] y' + \left[\frac{x^2 R(x)}{P(x)}\right] y = 0 \tag{2}$$

We will let

$$\frac{xQ(x)}{P(x)} = p(x) \qquad \text{and} \qquad \frac{x^2 R(x)}{P(x)} = q(x),$$

and furthermore we will assume $p(x)$ and $q(x)$ can be written as Taylor series centered at $x_0 = 0$. Thus we will assume that

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \qquad \text{and} \qquad q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Thus we can now write equation (2) in the following way.

$$x^2 y'' + x \left(\sum_{n=0}^{\infty} p_n x^n\right) y' + \left(\sum_{n=0}^{\infty} q_n x^n\right) y = 0 \tag{3}$$

We will look for a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \tag{4}$$

where $a_0 \neq 0$. Thus we differentiate the solution shown in (4) and substitute it into equation (3) to produce:

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \left(\sum_{n=0}^{\infty} p_n x^n\right) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} q_n x^n\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0.$$

Multiply the series for the second derivative of y by the x^2 expression and multiply the series for the first derivative of y by the x term. Now equation (2) can be written as

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \left(\sum_{n=0}^{\infty} p_n x^n\right) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} q_n x^n\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0.$$

As usual we must combine these several series into a single series using algebra and the procedure of re-indexing. We will accomplish this in several steps. First we write out the first several terms in each of the various power series.

$$\begin{aligned} 0 &= \left(r(r-1)a_0x^r + (1+r)ra_1x^{1+r} + (2+r)(1+r)a_2x^{2+r} + \cdots + (n+r)(n+r-1)a_nx^{n+r} + \cdots\right) + \\ &\left(p_0 + p_1x + p_2x^2 + \cdots + p_nx^n + \cdots\right)\left(ra_0x^r + (1+r)a_1x^{1+r} + (2+r)a_2x^{2+r} + \cdots + (n+r)a_nx^{n+r} + \cdots\right) \\ &+ \left(q_0 + q_1x + q_2x^2 + \cdots + q_nx^n + \cdots\right)\left(a_0x^r + a_1x^{1+r} + a_2x^{2+r} + \cdots + a_nx^{n+r} + \cdots\right) \end{aligned}$$

Now we multiply the appropriate series together and collect the like powers of x starting with the smallest power of x present, namely x^r. Thus the ordinary differential equation (2) can be written as

$$\begin{aligned} 0 &= \left(r(r-1)a_0 + rp_0a_0 + q_0a_0\right)x^r + \\ &\left((1+r)ra_1 + (1+r)p_0a_1 + q_0a_1 + rp_1a_0 + q_1a_0\right)x^{1+r} + \\ &\left((2+r)(1+r)a_2 + (2+r)p_0a_2 + q_0a_2 + (1+r)p_1a_1 + q_1a_1 + rp_2a_0 + q_2a_0\right)x^{2+r} \\ &+ \cdots + \\ &\left((n+r)(n+r-1)a_n + (n+r)p_0a_n + q_0a_n + (n+r-1)p_1a_{n-1} + q_1a_{n-1} + \right. \\ &\qquad \left.(n+r-2)p_2a_{n-2} + q_2a_{n-2} + \cdots + rp_na_0 + q_na_0\right)x^{n+r} + \cdots \end{aligned}$$

This can be simplified by grouping terms and factoring out common coefficients to the following,

$$\begin{aligned} 0 &= a_0\left(r(r-1) + p_0r + q_0\right)x^r + \\ &\left(a_1\left[(1+r)r + p_0(1+r) + q_0\right] + a_0\left[p_1r + q_1\right]\right)x^{1+r} + \\ &\left(a_2\left[(2+r)(1+r) + p_0(2+r) + q_0\right] + a_1\left[p_1(1+r) + q_1\right] + a_0\left[p_2r + q_2\right]\right)x^{2+r} \\ &+ \cdots + \\ &\left(a_n\left[(n+r)(n+r-1) + p_0(n+r) + q_0\right] + a_{n-1}\left[p_0(n+r) + q_0\right] + \right. \\ &\qquad \left.\cdots + a_1\left[p_{n-1}(1+r) + q_{n-1}\right] + a_0\left[p_nr + q_n\right]\right)x^{n+r} + \cdots \end{aligned}$$

Now if we define the quadratic function $F(r) = r(r-1) + p_0r + q_0$ then we can rewrite the previous equation as

$$\begin{aligned} 0 &= a_0F(r)x^r + \left(a_1F(1+r) + a_0\left[p_1r + q_1\right]\right)x^{1+r} + \left(a_2F(2+r) + a_1\left[p_1(1+r) + q_1\right] + a_0\left[p_2r + q_2\right]\right)x^{2+r} \\ &+ \cdots + \\ &\left(a_nF(n+r) + a_{n-1}\left[p_0(n+r) + q_0\right] + \cdots + a_1\left[p_{n-1}(1+r) + q_{n-1}\right] + a_0\left[p_nr + q_n\right]\right)x^{n+r} + \cdots \end{aligned}$$

Now if we use a finite sum within each expression involving a different power of x we can write the ordinary differential equation (2) as

$$\begin{aligned} 0 &= a_0F(r)x^r + \left(a_1F(1+r) + \sum_{k=0}^{0} a_k\left[p_{1-k}(k+r) + q_{1-k}\right]\right)x^{1+r} \\ &+ \left(a_2F(2+r) + \sum_{k=0}^{1} a_k\left[p_{2-k}(k+r) + q_{2-k}\right]\right)x^{2+r} \\ &+ \cdots + \left(a_nF(n+r) + \sum_{k=0}^{n-1} a_k\left[p_{n-k}(k+r) + q_{n-k}\right]\right)x^{n+r} + \cdots \end{aligned}$$

Now finally using a double summation we can write equation (2) as the following series.

$$a_0F(r)x^r + \sum_{n=1}^{\infty}\left(a_nF(n+r) + \sum_{k=0}^{n-1} a_k\left[p_{n-k}(k+r) + q_{n-k}\right]\right)x^{n+r} = 0 \qquad (5)$$

Remembering that we have assumed that $a_0 \neq 0$ and equating coefficients of powers of x on both sides of the equation we can conclude that

$$F(r) = r(r-1) + p_0r + q_0 = 0. \qquad (6)$$

Equation (6) is called the **indicial equation** of the ordinary differential equation. The indicial equation will have two roots called the **indicial roots** or the **exponents of singularity**. To keep our discussion simple, assume the roots are real numbers and label them r_1 and r_2 and arrange the labels so that $r_1 \leq r_2$. The coefficients of powers of x higher than r must also be zero, *i.e.*, for $n \geq 1$,

$$a_n F(n+r) + \sum_{k=0}^{n-1} a_k \left(p_{n-k}(k+r) + q_{n-k}\right) = 0$$

This allows us to derive the recurrence relation for $n \geq 1$

$$a_n(r) = -\frac{\sum_{k=0}^{n-1} a_k(r) \left(p_{n-k}(k+r) + q_{n-k}\right)}{F(n+r)} \tag{7}$$

Notice that a_n has been written as $a_n(r)$ to emphasize that the coefficients of the series solution depend on the value of the indicial root used for r. Thus in general $a_n(r_1) \neq a_n(r_2)$ and this is how two linearly independent solution of the second order equation (1) can be developed. When r_1 and r_2 are real and distinct roots which do not differ from one another by an integer amount, the general solution to equation (1) can be written as

$$y(x) = c_1 \sum_{n=0}^{\infty} a_n(r_1) x^{n+r_1} + c_2 \sum_{n=0}^{\infty} a_n(r_2) x^{n+r_2}$$

Complications can result in the following two cases:

1. $r_1 = r_2$, or
2. $r_1 + m = r_2$ for some positive integer m.

In the second case the recurrence relation (7) may be undefined when $r = r_1$ and $n = m$ due to a zero in the denominator of equation (7). However, we could use the recurrence relation with $r = r_2$ (the larger of the two indicial roots) without difficulty. Thus we can always find at least one series solution.

In the first case (referred to as the repeated indicial roots case), we still face the problem of finding a second linearly independent solution. Certainly one non-trivial solution to the ordinary differential equation is

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1) x^{n+r_1}.$$

If the roots of the indicial equation are repeated then the quadratic expression $F(r)$ must factor into $F(r) = (r - r_1)^2$. Hence thinking of the coefficients of the power series as functions of the variable r we can rewrite equation (5) as

$$a_0(r) F(r) x^r + \sum_{n=1}^{\infty} \left(a_n(r) F(n+r) + \sum_{k=0}^{n-1} a_k(r) \left[p_{n-k}(k+r) + q_{n-k}\right] \right) x^{n+r} = 0 \tag{8}$$

Keep in mind that $a_0(r) \equiv a_0$, an arbitrary non-zero constant. Now we will differentiate both sides of equation (8) with respect to r. This yields

$$\begin{aligned} 0 \;=\;& a_0 F'(r) x^r + a_0 F(r)(\ln x) x^r + \\ & \sum_{n=1}^{\infty} \left\{ a_n'(r) F(n+r) + a_n(r) F'(n+r) + \sum_{k=0}^{n-1} \left[a_k'(r)(p_{n-k}(k+r) + q_{n-k}) + a_k(r) p_{n-k}\right] \right\} x^{n+r} + \\ & \sum_{n=1}^{\infty} \left(a_n(r) F(n+r) + \sum_{k=0}^{n-1} a_k(r) \left[p_{n-k}(k+r) + q_{n-k}\right] \right) (\ln x) x^{n+r} \end{aligned}$$

using the fact that $\frac{d}{dr} x^r = (\ln x) x^r$. We now collect together all the terms involving $\ln x$ to produce

$$0 \;=\; a_0 F'(r) x^r +$$

$$
(\ln x)\left\{a_0F(r)x^r+\sum_{n=1}^{\infty}\left(a_n(r)F(n+r)+\sum_{k=0}^{n-1}a_k(r)\left[p_{n-k}(k+r)+q_{n-k}\right]\right)x^{n+r}\right\}+
$$

$$
\sum_{n=1}^{\infty}\left(a_n'(r)F(n+r)+a_n(r)F'(n+r)+\sum_{k=0}^{n-1}\left[a_k'(r)(p_{n-k}(k+r)+q_{n-k})+a_k(r)p_{n-k}\right]\right)x^{n+r}
$$

Now we must notice several things which occur if we replace r by r_1 in the equation above. First $F'(r) = 2(r-r_1)$ and thus $F'(r_1) = 0$. Second, the expression being multiplied by $\ln x$ is the differential equation (8) which is solved by $y_1(x)$ and thus the expression multiplying $\ln x$ is equal to 0 when $r = r_1$. Thus the previous equation can be rewritten as

$$
\sum_{n=1}^{\infty}\left(a_n'(r_1)F(n+r_1)+a_n(r_1)F'(n+r_1)+\sum_{k=0}^{n-1}\left[a_k'(r_1)(p_{n-k}(k+r_1)+q_{n-k})+a_k(r_1)p_{n-k}\right]\right)x^{n+r_1}=0
$$

To simplify this equation we will replace $a_n(r_1)$ with a_n and $a_n'(r_1)$ with b_n. We should note also that $F'(n+r_1) = 2n$. Since $a_0(r)$ is a constant, $b_0 = 0$. Thus we now have the following equation.

$$
\sum_{n=1}^{\infty}\left(b_nF(n+r_1)+2na_n+\sum_{k=0}^{n-1}\left[b_k(p_{n-k}(k+r_1)+q_{n-k})+a_kp_{n-k}\right]\right)x^{n+r_1}=0 \tag{9}
$$

To solve this equation we will again equate powers of x on both sides of the equation and thus we derive the following equation,

$$
b_nF(n+r_1)+2na_n+\sum_{k=0}^{n-1}\left(b_k[p_{n-k}(k+r_1)+q_{n-k}]+a_kp_{n-k}\right)=0
$$

which is true for $n \geq 1$. Now since $F(r)$ is zero only when $r = r_1$ we can solve the equation above for b_n. Thus we have derived a second recurrence relation for $n \geq 1$.

$$
b_n=-\frac{2na_n+\sum_{k=0}^{n-1}\left(b_k[p_{n-k}(k+r_1)+q_{n-k}]+a_kp_{n-k}\right)}{F(n+r_1)} \tag{10}
$$

We could also have derived this second recurrence relation by differentiating the recurrence relation in equation (7), setting r to r_1, and replacing $a_n(r_1)$ with a_n and $a_n'(r_1)$ with b_n.

So finally, the second linearly independent solution to equation (1) in the case of repeated indicial roots is given by

$$
\begin{aligned}
y_2(x) &= (\ln x)\sum_{n=0}^{\infty}a_nx^{n+r_1}+\sum_{n=1}^{\infty}b_nx^{n+r_1} \\
&= (\ln x)y_1(x)+\sum_{n=1}^{\infty}b_nx^{n+r_1} \qquad (11)
\end{aligned}
$$

If you do not find this differentiation argument convincing, you could take the *ad hoc* approach. Beginning with the ordinary differential equation (2) assume that a series solution has the form of the function $y_1(x)$ shown in equation (4). Differentiate this solution and substitute it into the ordinary differential equation. Simplify and combine all the series, solve for r_1 and for a_n for $n \geq 1$. Then assume the second linearly independent solution has the form of $y_2(x)$ shown in equation (11). Differentiate it and substitute into ordinary differential equation (2). Simplify and combine all the series and solve for b_n for $n \geq 1$. You will arrive at the same solution as outlined in this description.

Now if the exponents of singularity differ by a positive integer (*i.e.*, $r_1 = r_2 + m$) finding the second solution can be more involved. If the expression

$$
\sum_{k=0}^{n-1}a_k(r)(p_{n-k}(k+r)+q_{n-k})
$$

is divisible by $r - r_2 = r - r_1 - m$, then since $F(r)$ is also divisible by $r - r_2$, the term $a_m(r_2)$, generated by the recurrence relation in equation (7) is defined and hence so will the terms $a_{m+1}(r_2)$, $a_{m+2}(r_2)$, and so on. Thus the second linearly independent solution to the original ordinary differential equation will be of the form

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n,$$

with $a_0(r_2) = 1$.

If the expression $\sum_{k=0}^{n-1} a_k(r)(p_{n-k}(k+r) + q_{n-k})$ is not divisible by $r - r_2 = r - r_1 - m$, then let $a_0 \equiv a_0(r) = r - r_0$ and define

$$\psi(x,r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n. \tag{12}$$

By holding r fixed and differentiating the function with respect to x we see that

$$\begin{aligned}
&x^2\psi_{xx} + x(xp(x))\psi_x + x^2 q(x)\psi \\
&= \sum_{n=0}^{\infty} a_n(r)(n+r)(n+r-1)x^{n+r} + \left(\sum_{n=0}^{\infty} p_n x^n\right) \sum_{n=0}^{\infty} a_n(r)(n+r)x^{n+r} + \left(\sum_{n=0}^{\infty} q_n x^n\right) \sum_{n=0}^{\infty} a_n(r) x^{n+r} \\
&= (r-r_2)F(r)x^r + \sum_{n=1}^{\infty} \left(a_n(r)F(n+r) + \sum_{k=0}^{n-1} a_k(r)\left((r+k)p_{n-k} + q_{n-k}\right)\right) x^{n+r} \\
&= (r-r_2)F(r)x^r
\end{aligned}$$

since $a_n(r)$ will satisfy the recurrence relation in equation (7). Thus if we set $r = r_2$ the expression $(r - r_2)F(r)x^r = 0$ and we see that $\psi(x,r)$ solves the ordinary differential equation. If we now differentiate with respect to r, we obtain

$$\begin{aligned}
\frac{\partial}{\partial r}\left[x^2\psi_{xx} + x(xp(x))\psi_x + x^2 q(x)\psi\right] &= \frac{\partial}{\partial r}\left[(r-r_2)F(r)x^r\right] \\
x^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial \psi}{\partial r}\right) + x(xp(x))\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial r}\right) + x^2 q(x)\left(\frac{\partial \psi}{\partial r}\right) &= F(r)x^r + (r-r_2)F'(r)x^r + (r-r_2)F(r)(\ln x)x^r.
\end{aligned}$$

Thus we see that when $r = r_2$, the function $\left.\frac{\partial \psi}{\partial r}\right|_{r=r_2}$ satisfies the following equation.

$$x^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial \psi}{\partial r}\right) + x(xp(x))\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial r}\right) + x^2 q(x)\left.\left(\frac{\partial \psi}{\partial r}\right)\right|_{r=r_2} = 0$$

Therefore $\left.\frac{\partial \psi}{\partial r}\right|_{r=r_2}$ is a second linearly independent solution to the ordinary differential equation. From equation (12) we see that

$$\frac{\partial \psi}{\partial r} = (\ln x)x^r \sum_{n=0}^{\infty} a_n(r)x^n + x^r \sum_{n=0}^{\infty} a_n'(r)x^n,$$

and thus

$$y_2(x) = \left.\frac{\partial \psi}{\partial r}\right|_{r=r_2} = (\ln x)x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)x^n + x^{r_2} \sum_{n=0}^{\infty} a_n'(r_2)x^n.$$

The use of the recurrence relation 7 with $r = r_2$ may seem improper until we realize that by choosing $a_0 = r - r_2$ we have engineered a factor in the numerator of 7 which will cancel with the factor $r - r_2$ present in the quadratic indicial expression. Hence we have

$$\begin{aligned}
a_0(r_2) &= \left.(r-r_2)\right|_{r=r_2} = 0 \\
a_1(r_2) &= -\frac{a_0(r_2)(r_2(p_1+q_1)}{F(r_2+1} = 0
\end{aligned}$$

$$\vdots$$

$$a_{m-1}(r_2) = -\frac{\sum_{k=0}^{m-2} a_k(r_2)\left(p_{m-k-1}(k+r_2)+q_{m-k-1}\right)}{F(m+r_2-1)} = 0$$

Thus we have

$$\begin{aligned}
y_2(x) &= (\ln x)x^{r_2}\sum_{n=0}^{\infty} a_n(r_2)x^n + x^{r_2}\sum_{n=0}^{\infty} a_n'(r_2)x^n \\
&= (\ln x)x^{r_2}\sum_{n=m}^{\infty} a_n(r_2)x^n + x^{r_2}\sum_{n=0}^{\infty} a_n'(r_2)x^n \\
&= (\ln x)x^{r_1-m}\sum_{n=m}^{\infty} a_n(r_2)x^n + x^{r_2}\sum_{n=0}^{\infty} a_n'(r_2)x^n \\
&= (\ln x)x^{r_1}\sum_{n=m}^{\infty} a_n(r_2)x^{n-m} + x^{r_2}\sum_{n=0}^{\infty} a_n'(r_2)x^n \\
&= (\ln x)x^{r_1}\sum_{n+m=m}^{\infty} a_{n+m}(r_2)x^{n+m-m} + x^{r_2}\sum_{n=0}^{\infty} a_n'(r_2)x^n \\
&= (\ln x)x^{r_1}\sum_{n=0}^{\infty} a_{n+m}(r_2)x^n + x^{r_2}\sum_{n=0}^{\infty} a_n'(r_2)x^n
\end{aligned}$$

1.1 Indicial Roots Not Differing by an Integer

Consider the second order linear homogeneous differential equations,

$$4xy'' + 2y' + y = 0.$$

If we let $P(x) = x$, $Q(x) = 2$, and $R(x) = 1$, then we can easily see that $P(0) = 0$ (and thus $x_0 = 0$ is a singular point) and

$$\begin{aligned}
\lim_{x\to 0}\frac{xQ(x)}{P(x)} = \lim_{x\to 0}\frac{2x}{4x} = \frac{1}{2} &= p_0 \\
\lim_{x\to 0}\frac{x^2R(x)}{P(x)} = \lim_{x\to 0}\frac{x^2}{4x} = \lim_{x\to 0}\frac{x}{4} = 0 &= q_0,
\end{aligned}$$

and hence $x_0 = 0$ is a regular singular point.

The indicial equation will have the form,

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1) + \frac{r}{2} = r\left(r - \frac{1}{2}\right).$$

Thus the indicial roots are $r_1 = 0$ and $r_2 = 1/2$. In order to save ourselves some duplicated work, we can assume a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, differentiate, substitute into the ODE, re-index the series, combine them, and derive the recurrence relation once and then substitute the values we found for r.

$$\begin{aligned}
4x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} + 2\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} + \sum_{n=0}^{\infty}a_nx^{n+r} &= 0 \\
\sum_{n=0}^{\infty}4(n+r)(n+r-1)a_nx^{n+r-1} + \sum_{n=0}^{\infty}2(n+r)a_nx^{n+r-1} + \sum_{n=0}^{\infty}a_nx^{n+r} &= 0 \\
\sum_{n=0}^{\infty}2(n+r)a_n[2(n+r-1)+1]x^{n+r-1} + \sum_{n=0}^{\infty}a_nx^{n+r} &= 0
\end{aligned}$$

$$\begin{aligned}
\sum_{n+1=0}^{\infty} 2(n+r+1)a_{n+1}[2(n+r)+1]x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=-1}^{\infty} 2(n+r+1)(2n+2r+1)a_{n+1}x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
a_0 2r(2r-1)x^{r-1} + \sum_{n=0}^{\infty} 2(n+r+1)(2n+2r+1)a_{n+1}x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
a_0 2r(2r-1)x^{r-1} + \sum_{n=0}^{\infty} [2(n+r+1)(2n+2r+1)a_{n+1} + a_n]x^{n+r} &= 0
\end{aligned}$$

Thus we have derived the following recurrence relation.

$$a_{n+1} = -\frac{a_n}{2(n+r+1)(2n+2r+1)}, \qquad \text{for } n \geq 0$$

If we let r take on the value of $r_1 = 0$, the first indicial root, then we can specialize the recurrence relation to

$$a_{n+1} = -\frac{a_n}{2(n+1)(2n+1)}, \qquad \text{for } n \geq 0$$

To keep the discussion simple let $a_0 = 1$ (recall that a_0 can be any arbitrary non-zero value). We can find the first few coefficients in the power series solution:

$$\begin{aligned}
a_1 &= \frac{-a_0}{(2)(1)} = -\frac{1}{2!} \\
a_2 &= \frac{-a_1}{(4)(3)} = \frac{1}{4!} \\
a_3 &= \frac{-a_2}{(6)(5)} = -\frac{1}{6!} \\
&\vdots \\
a_n &= \frac{(-1)^n}{(2n)!}
\end{aligned}$$

Thus one solution to the ODE is

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{x})^{2n} = \cos\sqrt{x}.$$

Now if we let r take on the value $r_2 = 1/2$, the second indicial root, we obtain a second linearly independent solution. The recurrence relation becomes

$$a_{n+1} = -\frac{a_n}{2(n+1/2+1)(2n+2(1/2)+1)} = -\frac{a_n}{(2n+3)(2n+2)}, \qquad \text{for } n \geq 0$$

We can find the first few coefficients in the second power series solution:

$$\begin{aligned}
a_1 &= \frac{-a_0}{(3)(2)} = -\frac{1}{3!} \\
a_2 &= \frac{-a_1}{(5)(4)} = \frac{1}{5!} \\
a_3 &= \frac{-a_2}{(7)(6)} = -\frac{1}{7!} \\
&\vdots \\
a_n &= \frac{(-1)^n}{(2n+1)!}
\end{aligned}$$

Thus a second solution to the ODE is

$$y_2(x) = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \sqrt{x} (\sqrt{x})^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\sqrt{x})^{2n+1} = \sin \sqrt{x}.$$

Finally we see that the general solution to the ODE is

$$y(x) = c_1 \cos \sqrt{x} + c_2 \sin \sqrt{x}.$$

1.2 Repeated Indicial Roots Example

Consider the second order linear homogeneous differential equation,

$$x^2 y'' + (x^2 - x) y' + y = 0.$$

If we let $P(x) = x^2$, $Q(x) = x^2 - x$, and $R(x) = 1$, then we can easily see that $P(0) = 0$ (and thus $x_0 = 0$ is a singular point) and

$$\begin{aligned}
\lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} \frac{x(x^2 - x)}{x^2} = \lim_{x \to 0} (x - 1) = -1 &= p_0 \\
\lim_{x \to 0} \frac{x^2 R(x)}{P(x)} = \lim_{x \to 0} \frac{x^2 (1)}{x^2} = \lim_{x \to 0} 1 = 1 &= q_0,
\end{aligned}$$

and hence $x_0 = 0$ is a regular singular point.

The indicial equation will have the form,

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1) - r + 1 = (r-1)^2.$$

Thus the indicial roots are $r_1 = r_2 = 1$.

We want to look for a solution of the form $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$. Begin by differentiating this solution and substituting it into the differential equation. Then re-index and combine the series into a single series.

$$\begin{aligned}
x^2 \sum_{n=0}^{\infty} (n+1) n a_n x^{n-1} + (x^2 - x) \sum_{n=0}^{\infty} (n+1) a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} &= 0 \\
\sum_{n=0}^{\infty} (n+1) n a_n x^{n+1} + \sum_{n=0}^{\infty} (n+1) a_n x^{n+2} - \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} &= 0 \\
\sum_{n=1}^{\infty} (n+1) n a_n x^{n+1} + \sum_{n=0}^{\infty} (n+1) a_n x^{n+2} - \sum_{n=0}^{\infty} n a_n x^{n+1} &= 0 \\
\sum_{n=1}^{\infty} (n+1) n a_n x^{n+1} + \sum_{n=1}^{\infty} n a_{n-1} x^{n+1} - \sum_{n=1}^{\infty} n a_n x^{n+1} &= 0 \\
\sum_{n=1}^{\infty} (n a_n + a_{n-1}) n x^{n+1} &= 0
\end{aligned}$$

Thus we have derived the following recurrence relation.

$$a_n = -\frac{a_{n-1}}{n}, \qquad \text{for } n \geq 1$$

To keep the discussion simple let $a_0 = 1$ (recall that a_0 can be any arbitrary non-zero value). Then $a_1 = -1$, $a_2 = 1/2$, $a_3 = -1/3!$, and in general $a_n = (-1)^n / n!$. Thus the first solution has the form,

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1} = x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = x e^{-x}.$$

Now we must look for the second linearly independent solution. We should re-calculate the recurrence relation treating a_n as a function of the continuous variable r. This is most easily and accurately accomplished by starting from the beginning with the assumption that $y(x) = \sum_{n=0}^{\infty} a_n(r)x^{n+r}$. If we differentiate this solution and substitute into the ordinary differential equation we can derive the following.

$$\begin{aligned}
x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(r)x^{n+r-2} + (x^2-x)\sum_{n=0}^{\infty}(n+r)a_n(r)x^{n+r-1} + \sum_{n=0}^{\infty}a_n(r)x^{n+r} &= 0 \\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(r)x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n(r)x^{n+r+1} - \sum_{n=0}^{\infty}(n+r)a_n(r)x^{n+r} + \sum_{n=0}^{\infty}a_n(r)x^{n+r} &= 0 \\
\sum_{n=0}^{\infty}[(n+r)(n+r-1)-(n+r)+1]a_n(r)x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n(r)x^{n+r+1} &= 0 \\
\sum_{n=0}^{\infty}[(n+r)(n+r-1)-(n+r)+1]a_n(r)x^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1}(r)x^{n+r} &= 0 \\
(r-1)^2a_0x^r + \sum_{n=1}^{\infty}[(n+r-1)^2a_n(r) + (n+r-1)a_{n-1}(r)]x^{n+r} &= 0
\end{aligned}$$

Thus the re-derived recurrence relation for the a_n's is given by

$$a_n(r) = -\frac{a_{n-1}(r)}{n+r-1}.$$

Note that if we set $r = 1$ we obtain precisely the same recurrence relation as before. If we differentiate this recurrence relation with respect to r and then let $r = 1$ and assign $b_n = a_n'(1)$ we obtain a recurrence relation like the following.

$$b_n = -\frac{nb_{n-1} - a_{n-1}}{n^2} = \frac{1}{n}\left(\frac{a_{n-1}}{n} - b_{n-1}\right) = -\frac{1}{n}\left(a_n + b_{n-1}\right) = -\frac{1}{n}\left(\frac{(-1)^n a_0}{n!} + b_{n-1}\right)$$

As before we let $a_0 = 1$, then

$$\begin{aligned}
b_1 &= 1 \\
b_2 &= -(1 + \frac{1}{2!}) \\
b_3 &= 1 + \frac{1}{2!} + \frac{1}{3!} \\
&\vdots \\
b_n &= (-1)^{n+1}\sum_{k=1}^{n}\frac{1}{k!}
\end{aligned}$$

So we see that the second linearly independent solution is

$$y_2(x) = (\ln x)xe^{-x} + \sum_{n=1}^{\infty}\left((-1)^{n+1}\sum_{k=1}^{n}\frac{1}{k!}\right)x^{n+1}.$$

1.3 Indicial Roots Differing by an Integer Example

Consider the second order linear homogeneous differential equation,

$$xy'' + y = 0.$$

If we let $P(x) = x$, $Q(x) = 0$, and $R(x) = 1$, then we can easily see that $P(0) = 0$ (and thus $x_0 = 0$ is a singular point) and

$$\lim_{x\to 0}\frac{xQ(x)}{P(x)} = 0 \quad = \quad p_0$$

$$\lim_{x\to 0}\frac{x^2R(x)}{P(x)} = \lim_{x\to 0}\frac{x^2(1)}{x} = \lim_{x\to 0} x = 0 \quad = \quad q_0,$$

and hence $x_0 = 0$ is a regular singular point.

The indicial equation will have the form,

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1)$$

Thus the indicial roots are $r_1 = 1$ and $r_2 = 0$.

Our goal is to find two linearly independent solutions to the ODE. We can always find a series solution corresponding to the larger of the two indicial roots, so we could look for a solution of the form $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$. However, since we also want a second linearly independent solution we will follow the advice of the textbook authors who suggest that we obtain a recurrence relation involving the general formula $a_n(r)$ (see page 277). Thus we will look for a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

Begin by differentiating this solution and substituting it into the differential equation. Then re-index and combine the series into a single series.

$$\begin{aligned}
x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n+1=0}^{\infty}(n+r+1)(n+r)a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
r(r-1)a_0 x^{r-1} + \sum_{n=0}^{\infty}(n+r+1)(n+r)a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
r(r-1)a_0 x^{r-1} + \sum_{n=0}^{\infty}[(n+r+1)(n+r)a_{n+1} + a_n] x^{n+r} &= 0
\end{aligned}$$

Since $r = 0$ or $r = 1$, the first term above is zero and thus we have derived the following recurrence relation.

$$a_{n+1}(r) = -\frac{a_n(r)}{(n+r+1)(n+r)}, \qquad \text{for } n \geq 0$$

We begin by letting $r = 1$, the larger of the exponents of singularity. To keep the discussion simple let $a_0 = 1$ (recall that a_0 can be any arbitrary non-zero value). Then

$$\begin{aligned}
a_1 &= -\frac{a_0}{2\cdot 1} = -\frac{1}{2!1!} \\
a_2 &= -\frac{a_1}{3\cdot 2} = \frac{1}{3!2!} \\
a_3 &= -\frac{a_2}{4\cdot 3} = -\frac{1}{4!3!} \\
&\vdots \\
a_n &= \frac{(-1)^n}{(n+1)!n!}
\end{aligned}$$

Thus the first solution has the form,

$$y_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!n!}x^{n+1}.$$

According to Theorem 5.7.1 the second solution will have the form

$$y_2(x) = a y_1(x) \ln x + x^{r_2} \left[1 + \sum_{n=1}^{\infty} c_n(r_2) x^n \right]$$

where the coefficients $c_n(r_2)$ satisfy the formula

$$c_n(r_2) = \frac{d}{dr} \left[(r - r_2) a_n(r) \right] \Big|_{r=r_2}$$

for $n = 1, 2, \ldots$, and

$$a = \lim_{r \to r_2} (r - r_2) a_N(r)$$

where $N = r_1 - r_2$. Let's apply these formulas in this example. First $N = 1 = 1 - 0 = r_1 - r_2$, so

$$a = \lim_{r \to 0} r a_1(r) = \lim_{r \to 0} \frac{-a_0 r}{r(r+1)} = \lim_{r \to 0} \frac{-1}{r+1} = -1.$$

According to the quotient rule for derivatives

$$\begin{aligned} \frac{d}{dr}[(r - r_2) a_n(r)] &= \frac{d}{dr} \left[\frac{-r a_n(r)}{(n+r+1)(n+r)} \right] \\ &= \frac{-(a_n'(r) r + a_n(r))(n+r+1)(n+r) + a_n(r) r (2n + 2r + 1)}{(n+r+1)^2 (n+r)^2}. \end{aligned}$$

Thus when $r = r_2 = 0$,

$$c_n = \frac{-a_n}{n(n+1)} = \frac{(-1)^n}{(n+1)! n! (n+1) n}$$

and thus

$$y_2(x) = 1 - (\ln x) y_1(x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)! n! (n+1) n} x^n.$$