Background

- Previously we used iterated interpolation to approximate a function at a single point.
- Today we learn an iterated technique for building up the Lagrange interpolating polynomials.
Lagrange Interpolating Polynomial

Suppose $f(x)$ is a function and $P_n(x)$ is the $n$th Lagrange interpolating polynomial which agrees with $f(x)$ at the distinct points $\{x_0, x_1, \ldots, x_n\}$. 

Question: how can we find these constants?
Lagrange Interpolating Polynomial

Suppose $f(x)$ is a function and $P_n(x)$ is the $n$th Lagrange interpolating polynomial which agrees with $f(x)$ at the distinct points $\{x_0, x_1, \ldots, x_n\}$.

We can think of $P_n(x)$ as

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0) \cdots (x - x_{n-1})$$

$$= a_0 + \sum_{i=1}^{n} a_i \prod_{j=0}^{i-1} (x - x_j)$$

for an appropriate choice of constants $a_0, a_1, \ldots, a_n$.

**Question:** how can we find these constants?
Evaluation of $P_n(x)$

\[ P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0) \cdots (x - x_{n-1}) \]

- If $x = x_0$ then $P_n(x_0) = f(x_0) = a_0$. 
Evaluation of $P_n(x)$

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$$

- If $x = x_0$ then $P_n(x_0) = f(x_0) = a_0$.
- If $x = x_1$ then $P_n(x_1) = f(x_1)$ and

\[
\begin{align*}
P_n(x_1) &= a_0 + a_1(x_1 - x_0) \\
f(x_1) &= f(x_0) + a_1(x_1 - x_0) \\
a_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0}
\end{align*}
\]
Evaluation of $P_n(x)$

\[
P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0) \cdots (x - x_{n-1})
\]

- If $x = x_0$ then $P_n(x_0) = f(x_0) = a_0$.
- If $x = x_1$ then $P_n(x_1) = f(x_1)$ and

\[
P_n(x_1) = a_0 + a_1(x_1 - x_0)
\]
\[
f(x_1) = f(x_0) + a_1(x_1 - x_0)
\]
\[
a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}
\]

- and so on.
Find $a_2$

\[ P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \]

\[ f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \]

\[ a_2(x_2 - x_0)(x_2 - x_1) = f(x_2) - f(x_0) - \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) \]

\[ a_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \]
Divided Difference Notation (1 of 2)

- Denote the **zeroth divided difference** of $f$ with respect to $x_i$ by
  \[ f[x_i] = f(x_i). \]

- Denote the **first divided difference** of $f$ with respect to $x_i$ and $x_{i+1}$ by
  \[ f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}. \]

- Denote the **second divided difference** of $f$ with respect to $x_i$, $x_{i+1}$, and $x_{i+2}$ by
  \[ f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}. \]
Proceeding recursively,

- Denote the $k$th divided difference of $f$ with respect to $x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k}$ by

$$f[x_i, x_{i+1}, \ldots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}.$$ 

- Finally, denote the $n$th divided difference of $f$ with respect to $x_0, x_1, x_2, \ldots, x_n$ by

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$
Summary and Connections

Recall that

\[ P_n(x) = a_0 + \sum_{k=1}^{n} a_k \prod_{j=0}^{k-1} (x - x_j). \]
Summary and Connections

Recall that

\[ P_n(x) = a_0 + \sum_{k=1}^{n} a_k \prod_{j=0}^{k-1} (x - x_j). \]

Using the divided difference notation we see that

\[ a_0 = f[x_0] \]
\[ a_1 = f[x_0, x_1] \]
\[ a_2 = f[x_0, x_1, x_2] \]
\[ \vdots \]
\[ a_n = f[x_0, x_1, x_2, \ldots, x_n], \quad \text{and thus} \]

\[ P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j). \]

This is called Newton’s interpolatory divided difference formula.
### Table Format

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>First</th>
<th>Second</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>$f[x_0]$</td>
<td>$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$</td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>$f[x_1]$</td>
<td>$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$</td>
<td>$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$f[x_2]$</td>
<td>$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$</td>
<td>$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$f[x_3]$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The third divided difference is not shown.
Divided Difference Algorithm

INPUT nodes \{ (x_0, f(x_0)), \ldots, (x_n, f(x_n)) \}

STEP 1 For \( i = 0, 1, \ldots, n \) set \( F_{i, 0} = f(x_i) \).

STEP 2 For \( i = 1, 2, \ldots, n \)
   For \( j = 1, 2, \ldots, i \) set
   \[
   F_{i, j} = \frac{F_{i, j-1} - F_{i-1, j-1}}{x_i - x_{i-j}}
   \]

STEP 3 OUTPUT \( F_{0,0}, F_{1,1}, \ldots, F_{n,n} \). STOP.
Example (1 of 2)

Complete the divided difference table and construct the interpolating polynomial.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>First</th>
<th>Second</th>
<th>Third</th>
<th>Fourth</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2</td>
<td>22.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>8.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.7</td>
<td>17.8</td>
<td>2.11765</td>
<td>2.85561</td>
<td>–0.52748</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>14.2</td>
<td>6.34211</td>
<td>2.01165</td>
<td>0.0865307</td>
<td>0.255838</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.8</td>
<td>38.3</td>
<td>16.75</td>
<td>2.26259</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.6</td>
<td>51.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example (2 of 2)

\[ P_4(x) = 22.0 + 8.4(x - 3.2) + 2.85561(x - 3.2)(x - 2.7) \\
- 0.52748(x - 3.2)(x - 2.7)(x - 1.0) \\
+ 0.255838(x - 3.2)(x - 2.7)(x - 1.0)(x - 4.8) \\
= 34.96 - 36.1836x + 18.6885x^2 - 3.52078x^3 \\
+ 0.255838x^4 \]
Implications of the Mean Value Theorem

\[ f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i} = f'(z) \]

for some \( z \) between \( x_i \) and \( x_j \) according to the MVT.
Implications of the Mean Value Theorem

\[ f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i} = f'(z) \]

for some \( z \) between \( x_i \) and \( x_j \) according to the MVT.

This can be generalized.

**Theorem**

Suppose \( f \in C^n[a, b] \) and \( x_0, x_1, \ldots, x_n \) are distinct numbers in \([a, b]\). There exists \( z \in (a, b) \) such that

\[ f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(z)}{n!}. \]
Proof

- Define $g(x) = f(x) - P_n(x)$.
- Since $f(x_i) = P_n(x_i)$ for $i = 0, 1, \ldots, n$, then function $g$ has $n + 1$ distinct roots in $[a, b]$.
- According to the Generalized Rolle’s Theorem, $g^{(n)}(z) = 0$ for some $z \in (a, b)$.

\[
0 = g^{(n)}(z) \\
= f^{(n)}(z) - P_n^{(n)}(z) \\
P_n^{(n)}(z) = f^{(n)}(z) \\
n! f[x_0, x_1, \ldots, x_n] = f^{(n)}(z)
\]
Remarks

- The coordinates of the nodes $x_0, x_1, \ldots, x_n$ need not be in ascending order.
- The spacing between the nodes $\Delta x_i = x_{i+1} - x_i$ need not be uniform.
Remarks

- The coordinates of the nodes $x_0, x_1, \ldots, x_n$ need not be in ascending order.
- The spacing between the nodes $\Delta x_i = x_{i+1} - x_i$ need not be uniform.

However, if the nodes are in ascending order and the spacing between nodes is uniform, we can modify Newton’s divided difference formula.
Suppose $x_{i+1} - x_i = h > 0$ for $i = 0, 1, \ldots, n - 1$, then

- For any $x$ there exists $s$ such that $x = x_0 + sh$.
- In particular $x_i = x_0 + ih$ for $i = 0, 1, \ldots, n$.
- For $i = 0, 1, \ldots, n$ the difference

\[
x - x_i = (x_0 + sh) - x_i = (x_0 + sh) - (x_0 + ih) = (s - i)h.
\]
Forward Differences (1 of 4)

Suppose \( x_{i+1} - x_i = h > 0 \) for \( i = 0, 1, \ldots, n - 1 \), then

- For any \( x \) there exists \( s \) such that \( x = x_0 + sh \).
- In particular \( x_i = x_0 + ih \) for \( i = 0, 1, \ldots, n \).
- For \( i = 0, 1, \ldots, n \) the difference

\[
x - x_i = (x_0 + sh) - x_i = (x_0 + sh) - (x_0 + ih) = (s - i)h.
\]

\[
P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j)
\]

\[
P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (x_0 + sh - x_j)
\]

\[
= f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} ((s - j)h)
\]

\[
= f[x_0] + \sum_{k=1}^{n} h^k f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (s - j)
\]
Forward Differences (2 of 4)

Using the binomial coefficient notation

\[
\binom{s}{k} = \frac{s!}{(s-k)!k!} = \frac{s(s-1) \cdots (s-k+1)}{k!}
\]

\[
s(s-1) \cdots (s-k+1) = \prod_{j=0}^{k-1} (s-j) = k! \binom{s}{k},
\]

we can write

\[
P_n(x) = f[x_0] + \sum_{k=1}^{n} h^k f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (s-j)
\]

\[
= f[x_0] + \sum_{k=1}^{n} h^k k! \binom{s}{k} f[x_0, \ldots, x_k].
\]
Forward Differences (3 of 4)

Recalling Aitken’s $\Delta^2$ notation we may write

\[
f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h} \Delta f(x_0)
\]

\[
f[x_0, x_1, x_2] = \frac{f[x_2, x_1] - f[x_0, x_1]}{x_2 - x_0} = \frac{1}{2h} \left( \frac{1}{h} \Delta f(x_1) - \frac{1}{h} \Delta f(x_0) \right) = \frac{1}{2h^2} \Delta^2 f(x_0)
\]

\[\vdots\]

\[
f[x_0, x_1, \ldots, x_k] = \frac{1}{k! h^k} \Delta^k f(x_0).
\]
Finally, we may write the **Newton Forward-Difference Formula**:

\[
P_n(x) = f[x_0] + \sum_{k=1}^{n} h^k k! \binom{s}{k} f[x_0, \ldots, x_k]
\]

\[
= f[x_0] + \sum_{k=1}^{n} h^k k! \binom{s}{k} \frac{1}{k!} h^k \Delta^k f(x_0)
\]

\[
= f[x_0] + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0)
\]
Forward-differences on the nodes

\[ x_0 < x_1 < \cdots < x_{n-1} < x_n \]

are useful when \( x \) is nearer to \( x_0 \) than to \( x_n \) since generally \( f(x_0) \) will be closer to \( f(x) \) than will \( f(x_n) \).
Forward-differences on the nodes

\[ x_0 < x_1 < \cdots < x_{n-1} < x_n \]

are useful when \( x \) is nearer to \( x_0 \) than to \( x_n \) since generally \( f(x_0) \) will be closer to \( f(x) \) than will \( f(x_n) \).

If we need to approximate \( f \) at \( x \) near \( x_n \) then we should reorder the nodes as

\[ x_n > x_{n-1} > \cdots > x_1 > x_0. \]

The interpolating polynomial becomes

\[
P_n(x) = f[x_n] + \sum_{i=1}^{n} f[x_n, \ldots, x_{n-i}](x - x_n) \cdots (x - x_{n-i+1}).
\]
Definition
Given the sequence \( \{p_n\}_{n=0}^{\infty} \) we define the \textbf{backward difference} \( \nabla p_n \) as

\[
\nabla p_n = p_n - p_{n-1}, \quad \text{for } n \geq 1.
\]

For \( k \geq 2 \) we define the \( k \)th order backward difference as

\[
\nabla^k p_n = \nabla (\nabla^{k-1} p_n).
\]
Using the backward difference notation we may write

\[ f[x_n, x_{n-1}] = \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n} \]

\[ = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \]

\[ = \frac{1}{h} \nabla f(x_n) \]

\[ f[x_n, x_{n-1}, x_{n-2}] = \frac{1}{2h^2} \nabla^2 f(x_n) \]

\[ \vdots \]

\[ f[x_n, x_{n-1}, \ldots, x_{n-k}] = \frac{1}{k! h^k} \nabla^k f(x_n). \]
Writing \( x = x_n + sh \) where \( s < 0 \) and \( x - x_i = (s + n - i)h \) then the interpolating polynomial can be written as

\[
P_n(x) = f[x_n] + \sum_{i=1}^{n} f[x_n, \ldots, x_{n-i}](x - x_n) \cdots (x - x_{n-i+1})
\]

\[
= f[x_n] + \sum_{i=1}^{n} h^i s(s + 1) \cdots (s + n - i) f[x_n, \ldots, x_{n-i}]
\]

\[
= f[x_n] + \sum_{i=1}^{n} \frac{s(s + 1) \cdots (s + n - i)}{i!} \nabla^i f[x_n].
\]
Since $s < 0$ we must modify the binomial coefficient notation.

$$\binom{-s}{k} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!} = (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!}$$
Since $s < 0$ we must modify the binomial coefficient notation.

\[
\binom{-s}{k} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!} = (-1)^k \frac{s(s+1)\cdots(s+k-1)}{k!}
\]

Then we may write the interpolating polynomial as

\[
P_n(x) = f[x_n] + \sum_{i=1}^{n} \frac{s(s+1)\cdots(s+n-i)}{i!} \nabla^i f[x_n]
\]

\[
= f[x_n] + \sum_{i=1}^{n} (-1)^i \binom{-s}{i} \nabla^i f[x_n]
\]

This is known as the Newton backward-difference formula.
Suppose we create **forward** and **backward** difference interpolating polynomials for \( f(x) = \cos x \) using nodes \( x_i = 0.2(i + 1) \) for \( i = 0, 1, 2, 3 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \cos x )</th>
<th><strong>First</strong></th>
<th><strong>Second</strong></th>
<th><strong>Third</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.980067</td>
<td>(-0.295028)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.921061</td>
<td>(-0.478627)</td>
<td>(-0.458997)</td>
<td>0.0795056</td>
</tr>
<tr>
<td>0.6</td>
<td>0.825336</td>
<td>(-0.643145)</td>
<td>(-0.411294)</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.696707</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example (2 of 4)

Forward divided difference:

\[ P_3(x) = 0.998536 + 0.015353x - 0.554404x^2 + 0.0795056x^3 \]

Backward divided difference:

\[ P_3(x) = 0.998537 + 0.0153524x - 0.554404x^2 + 0.0795056x^3 \]
Example (3 of 4)
Homework

- Read Section 3.3.
- Exercises: 1a, 3a, 5a, 13, 17