Motivation

Several root-finding algorithms operate by repeatedly evaluating a function until its value does not change (or changes by a suitably small amount).
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Definition

A **fixed-point** for a function $f$ is a number $p$ such that $f(p) = p$. 
Equivalence

The process of root-finding and the process of finding fixed points are equivalent in the following sense.

- Suppose $g(x)$ is a function with a root at $x = p$, then $f(x) = g(x) + x$ has a fixed point at $x = p$.
- Suppose $f(x)$ is a function with a fixed point at $x = p$, then $g(x) = x - f(x)$ has a root at $x = p$. 
Find the fixed points (if any) of the following functions.

- $f(x) = x$
- $f(x) = 2x$
- $f(x) = x - \sin \pi x$
Graphical Interpretation
Existence and Uniqueness

Theorem (Brouwer)
Suppose $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. Then

1. $g$ has a fixed point in $[a, b]$ (existence), and
2. if, in addition, $g'(x)$ exists on $(a, b)$ and if there exists a constant $0 < k < 1$ such that

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b),$$

then there is exactly one fixed point in $[a, b]$ (uniqueness).
**Graphical Interpretation**

**Note:** $g(a) \geq a$ and $g(b) \leq b$. If $g(a) = a$ or if $g(b) = b$ then $g$ has a fixed point in $[a, b]$. 
Suppose \( g(a) > a \) and \( g(b) < b \).

Define \( h(x) = g(x) - x \). The function \( h \in C[a, b] \).

\( h(b) < 0 < h(a) \) so by the Intermediate Value Theorem there exists \( p \in (a, b) \) such that \( h(p) = 0 \)

\[
\begin{align*}
0 &= h(p) \\
&= g(p) - p \\
g(p) &= p
\end{align*}
\]
Now suppose $g'(x)$ exists on $(a, b)$ and there exists a constant $0 < k < 1$ such that $|g'(x)| \leq k$ for all $x \in (a, b)$. 
Proof (2 of 2)

Now suppose $g'(x)$ exists on $(a, b)$ and there exists a constant $0 < k < 1$ such that $|g'(x)| \leq k$ for all $x \in (a, b)$.

For the purposes of contradiction suppose $g$ has two fixed points $p$ and $q$ with $p \neq q$. 
Now suppose \( g'(x) \) exists on \((a, b)\) and there exists a constant \( 0 < k < 1 \) such that \( |g'(x)| \leq k \) for all \( x \in (a, b) \).

For the purposes of contradiction suppose \( g \) has two fixed points \( p \) and \( q \) with \( p \neq q \).

According to the Mean Value Theorem there exists \( c \) between \( p \) and \( q \) for which

\[
\left| \frac{g(p) - g(q)}{p - q} \right| = \frac{p - q}{p - q} = 1 = |g'(c)|.
\]

This contradicts the assumption that \( |g'(x)| \leq k < 1 \) for all \( x \in (a, b) \).
Example

Show that $g(x) = 2^{-x}$ has a unique fixed point on $[0, 1]$. 
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- $g \in \mathcal{C}[0, 1]$ and $g'(x) = -(\ln 2)2^{-x} < 0$ which implies $g$ is monotone decreasing.

- $g(0) = 1$ and $g(1) = 1/2 > 0$. Since $g$ is decreasing then $g(x) \in [0, 1]$ for all $x \in [0, 1]$.

- By the first conclusion of the Brouwer Theorem, $g$ has a fixed point in $[0, 1]$. 
Example

Show that \( g(x) = 2^{-x} \) has a unique fixed point on \([0, 1]\).

\(- g \in C[0, 1] \) and \( g'(x) = -(\ln 2)2^{-x} < 0 \) which implies \( g \) is monotone decreasing.

\(- g(0) = 1 \) and \( g(1) = 1/2 > 0 \). Since \( g \) is decreasing then \( g(x) \in [0, 1] \) for all \( x \in [0, 1] \).

\(- \) By the first conclusion of the Brouwer Theorem, \( g \) has a fixed point in \([0, 1]\).

\(- |g'(x)| = |-(\ln 2)2^{-x}| \leq \ln 2 < 1 \) for \( x \in (0, 1) \).

\(- \) By the second conclusion of the Brouwer Theorem, the fixed point is unique.
Let $p$ be the unknown fixed point of $g(x) = 2^{-x}$ and suppose $\hat{p}$ is an approximation to $p$.

\[
\frac{g(p) - g(\hat{p})}{p - \hat{p}} = g'(y) \quad \text{(for some $y$ between $p$ and $\hat{p}$)}
\]

Conclusion: $2^{-\hat{p}}$ is a better approximation to $p$ than $\hat{p}$ was.
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\[
\frac{g(p) - g(\hat{p})}{p - \hat{p}} = g'(y) \quad (\text{for some } y \text{ between } p \text{ and } \hat{p})
\]

\[
|g(p) - g(\hat{p})| = |g'(y)| |p - \hat{p}|
\]

\[
|p - 2^{-\hat{p}}| = \left| - (\ln 2)2^{-y} \right| |p - \hat{p}|
\]

\[
< 1
\]

Conclusion: $2^{-\hat{p}}$ is a better approximation to $p$ than $\hat{p}$ was.
Let \( p \) be the unknown fixed point of \( g(x) = 2^{-x} \) and suppose \( \hat{p} \) is an approximation to \( p \).

\[
\frac{g(p) - g(\hat{p})}{p - \hat{p}} = g'(y) \quad \text{(for some } y \text{ between } p \text{ and } \hat{p})
\]

\[
|g(p) - g(\hat{p})| = |g'(y)| \cdot |p - \hat{p}|
\]

\[
|p - 2^{-\hat{p}}| = \left| - (\ln 2)2^{-y} \right| \cdot |p - \hat{p}| < 1
\]

\[
|p - 2^{-\hat{p}}| < |p - \hat{p}|
\]

Conclusion: \( 2^{-\hat{p}} \) is a better approximation to \( p \) than \( \hat{p} \) was.
Let $p$ be the unknown fixed point of $g(x) = 2^{-x}$ and suppose $\hat{p}$ is an approximation to $p$.

\[
\frac{g(p) - g(\hat{p})}{p - \hat{p}} = g'(y) \quad (\text{for some } y \text{ between } p \text{ and } \hat{p})
\]

\[
|g(p) - g(\hat{p})| = |g'(y)||p - \hat{p}|
\]

\[
|p - 2^{-\hat{p}}| = \left| - (\ln 2)2^{-y} \right||p - \hat{p}| < 1
\]

\[
|p - 2^{-\hat{p}}| < |p - \hat{p}|
\]

**Conclusion:** $2^{-\hat{p}}$ is a better approximation to $p$ than $\hat{p}$ was.
Let $g(x) = 2^{-x}$ and let $x_0 = 1/2$. Given $x_{n-1}$ then define $x_n = g(x_{n-1})$.

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Cobweb Diagram
Fixed-Point Iteration

Given $g(x)$:

**INPUT**  $p_0$, tolerance $\epsilon$, maximum iterations $N$

**STEP 1**  Set $i = 1$.

**STEP 2**  While $i \leq N$ do STEPS 3–6.

**STEP 3**  Set $p = g(p_0)$.

**STEP 4**  If $|p - p_0| < \epsilon$ then OUTPUT $p$; STOP.

**STEP 5**  Set $i = i + 1$.

**STEP 6**  Set $p_0 = p$.

**STEP 7**  OUTPUT “The method failed after $N$ iterations.”; STOP.
Example (1 of 2)

Approximate a solution to \( x^3 - x - 1 = 0 \) on \([1, 2]\) using fixed point iteration.
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Approximate a solution to \( x^3 - x - 1 = 0 \) on \([1, 2]\) using fixed point iteration.

If we let \( g(x) = x^3 - 1 \) then finding a fixed point of \( g \) is equivalent to finding a root of the original equation.
Example (1 of 2)

Approximate a solution to $x^3 - x - 1 = 0$ on $[1, 2]$ using fixed point iteration.

If we let $g(x) = x^3 - 1$ then finding a fixed point of $g$ is equivalent to finding a root of the original equation.

However; $g(x)$ maps the interval $[1, 2]$ to the interval $[1, 7]$. 
Approximate a solution to $x^3 - x - 1 = 0$ on $[1, 2]$ using fixed point iteration.

If we let $g(x) = x^3 - 1$ then finding a fixed point of $g$ is equivalent to finding a root of the original equation.

However; $g(x)$ maps the interval $[1, 2]$ to the interval $[1, 7]$.

Can you find another function $g$ which maps $[1, 2]$ into itself?
Example (2 of 2)

\[ x^3 - x - 1 = 0 \]
\[ x^3 = x + 1 \]
\[ x = \sqrt[3]{x + 1} = g(x) \]

Function \( g \) maps \([1, 2]\) into \([1, 2]\). Starting with \( p_0 = 1 \) the sequence of approximations to the fixed point is

\( \{1.0, 1.25992, 1.31229, 1.32235, 1.32427, 1.32463, 1.32470\} \)
Fixed-Point Theorem

Theorem (Fixed-Point)
Suppose $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. Furthermore suppose that $g'$ exists on $(a, b)$ and that there exists a constant $0 < k < 1$ such that

$$|g'(x)| \leq k, \text{ for } x \in [a, b].$$

Then for any $p_0 \in [a, b]$ the sequence defined by

$$p_n = g(p_{n-1}), \text{ for } n \geq 1$$

converges to the unique fixed point $p$ in $[a, b]$. 
According to Brouwer’s Theorem the unique fixed point $p$ exists in $[a, b]$.

The sequence $\{p_n\}_{n=0}^{\infty}$ has the property that $p_n \in [a, b]$ for all $n$.

By the MVT

\[
\left| \frac{g(p_0) - g(p)}{p_0 - p} \right| = |g'(\xi_0)| \quad \text{(with } \xi_0 \in [a, b])
\]

| $p_1 - p$ | $= |g'(\xi_0)\cdot|p_0 - p|$ |
| $p_1 - p$ | $\leq k\cdot|p_0 - p|$.
Proof (2 of 2)

▶ Suppose $|p_n - p| \leq k^n |p_0 - p|$ for some $n \geq 1$.
▶ By the MVT

$$\left| \frac{g(p_n) - g(p)}{p_n - p} \right| = |g'(\xi_n)| \quad (\text{with } \xi_n \in [a, b])$$

$$|p_{n+1} - p| = |g'(\xi_n)| |p_n - p|$$

$$|p_{n+1} - p| \leq k |p_n - p|$$

$$|p_{n+1} - p| \leq k^{n+1} |p_0 - p|.$$ 

▶ Since $0 < k < 1$, the $\lim_{n \to \infty} |p_n - p| = 0$. 
Error Estimate

While the Fixed-Point Theorem justifies that the algorithm will converge to a fixed-point/solution of the function/equation, it does not tell us anything directly about the error present in each stage of the algorithm.

\[ |p_n - p| \leq k^n \max \{ p_0 - a, b - p_0 \} \]

and

\[ |p_n - p| \leq k^n \frac{1}{1 - k} |p_1 - p|, \quad \text{for } n \geq 1 \]
Error Estimate

While the Fixed-Point Theorem justifies that the algorithm will converge to a fixed-point/solution of the function/equation, it does not tell us anything directly about the error present in each stage of the algorithm.

Corollary

*If $g$ satisfies the hypotheses of the Fixed-Point Theorem, then bounds for the error involved in using $p_n$ to approximate $p$ are*

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

*and*

$$|p_n - p| \leq \frac{k^n}{1 - k} |p_1 - p|, \quad \text{for } n \geq 1.$$
Proof (1 of 2)

According to the Fixed-Point Theorem

$$|p_n - p| \leq k^n |p_0 - p| \leq k^n \max\{p_0 - a, b - p_0\}.$$
Proof (1 of 2)

According to the Fixed-Point Theorem

\[ |p_n - p| \leq k^n |p_0 - p| \leq k^n \max\{p_0 - a, b - p_0\}. \]

Using the MVT again we see that

\[ |p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \]
\[ \leq k|p_n - p_{n-1}| \]
\[ \leq k^2|p_{n-1} - p_{n-2}| \]
\[ \vdots \]
\[ \leq k^n|p_1 - p_0|. \]
Proof (2 of 2)

Assume $1 \leq n < m$ then

$$|p_m - p_n| \leq |p_m - p_{m-1} + p_{m-1} - \cdots + p_{n+1} - p_n|$$

$$\leq |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \cdots + |p_{n+1} - p_n|$$

$$\leq k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + \cdots + k^n|p_1 - p_0|$$

$$= k^n(1 + k + k^2 + \cdots + k^{m-n-1})|p_1 - p_0|$$

$$\leq k^n \left( \sum_{i=0}^{\infty} k^i \right) |p_1 - p_0|$$

$$= \frac{k^n}{1 - k} |p_1 - p_0|$$
Consider the equation \( x^3 + 4x^2 - 10 = 0 \).

- Develop a fixed point function to approximate a solution to this equation on the interval \([1, 2]\).
- Determine the number of iterations necessary to estimate the solution to within \(10^{-6}\).
- Estimate the solution.
Solution

Among other possibilities:

\[ 0 = x^3 + 4x^2 - 10 \]
\[ x^2(x + 4) = 10 \]
\[ x = \sqrt{\frac{10}{x + 4}} = g(x). \]
Solution

- Among other possibilities:

\[ 0 = x^3 + 4x^2 - 10 \]
\[ x^2(x + 4) = 10 \]
\[ x = \sqrt{\frac{10}{x + 4}} = g(x). \]

- Upon taking the derivative of \( g(x) \) we see that on \([1, 2]\),

\[ |g'(x)| = \left| \frac{-5}{\sqrt{10(x + 4)^{3/2}}} \right| \leq \frac{5}{\sqrt{10} (5)^{3/2}} < 0.1415 = k. \]

If \( p_0 = 1 \) then \( p_1 = \sqrt{2} \) and

\[ |p - p_n| \leq \frac{k^n}{1 - k} |p_1 - p_0| = \frac{0.1415^n}{1 - 0.1415} |\sqrt{2} - 1| < 10^{-6} \]
\[ n \geq 7 \]
Solution

Among other possibilities:

\[ 0 = x^3 + 4x^2 - 10, \]
\[ x^2(x + 4) = 10. \]
\[ x = \sqrt{\frac{10}{x + 4}} = g(x). \]

Upon taking the derivative of \( g(x) \) we see that on \([1, 2]\),

\[ |g'(x)| = \left| \frac{-5}{\sqrt{10(x + 4)^{3/2}}} \right| \leq \frac{5}{\sqrt{10}(5)^{3/2}} < 0.1415 = k. \]

If \( p_0 = 1 \) then \( p_1 = \sqrt{2} \) and

\[ |p - p_n| \leq \frac{k^n}{1 - k} |p_1 - p_0| = \frac{0.1415^n}{1 - 0.1415} |\sqrt{2} - 1| < 10^{-6}. \]

\[ n \geq 7 \]

\[ p_7 \approx 1.365230 \]
Challenge

For a given equation $f(x) = 0$, find a fixed point function which satisfies the conditions of the Fixed-Point Theorem.
Homework

- Read Section 2.2.
- Exercises: 1, 2, 5, 9, 15, 18, 20