Interpolation and the Lagrange Polynomial
MATH 375

J. Robert Buchanan

Department of Mathematics

Fall 2015
We often choose polynomials to approximate other classes of functions.

**Theorem (Weierstrass Approximation Theorem)**

If \( f \in C[a, b] \) and \( \epsilon > 0 \) then there exists a polynomial \( P \) such that

\[
|f(x) - P(x)| < \epsilon, \quad \text{for all } x \in [a, b].
\]
We often choose **polynomials** to approximate other classes of functions.

**Theorem (Weierstrass Approximation Theorem)**  
If $f \in C[a, b]$ and $\epsilon > 0$ then there exists a polynomial $P$ such that  
\[
|f(x) - P(x)| < \epsilon, \quad \text{for all } x \in [a, b].
\]

- We have used **Taylor polynomials** to approximate functions.  
  \[
P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k
\]
- Away from $x_0$ the approximation may be very poor.
Using the two-point formula for a line we determine the equation of the line to be

\[ y = f(x_0) + \left( \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) (x - x_0). \]
We can take another approach to finding the linear approximation. Define

\[ L_0(x) = \frac{x - x_1}{x_0 - x_1}, \]

\[ L_1(x) = \frac{x - x_0}{x_1 - x_0}, \]

If \( f(x) \) is any function then

\[ P(x) = f(x_0) L_0(x) + f(x_1) L_1(x) \]

is a linear function which matches \( f(x) \) at \( x = x_0 \) and \( x = x_1 \).

We want to extend this idea to higher order polynomials.
We can take another approach to finding the linear approximation. Define

- $L_0(x) = \frac{x - x_1}{x_0 - x_1}$, then $L_0(x_0) = 1$ and $L_0(x_1) = 0$.
- $L_1(x) = \frac{x - x_0}{x_1 - x_0}$,
We can take another approach to finding the linear approximation. Define

- $L_0(x) = \frac{x - x_1}{x_0 - x_1}$, then $L_0(x_0) = 1$ and $L_0(x_1) = 0$.

- $L_1(x) = \frac{x - x_0}{x_1 - x_0}$, then $L_1(x_0) = 0$ and $L_1(x_1) = 1$. 

If $f(x)$ is any function then $P(x) = f(x_0)L_0(x) + f(x_1)L_1(x)$ is a linear function which matches $f(x)$ at $x = x_0$ and $x = x_1$.

We want to extend this idea to higher order polynomials.
We can take another approach to finding the linear approximation. Define

\[ L_0(x) = \frac{x - x_1}{x_0 - x_1}, \text{ then } L_0(x_0) = 1 \text{ and } L_0(x_1) = 0. \]

\[ L_1(x) = \frac{x - x_0}{x_1 - x_0}, \text{ then } L_1(x_0) = 0 \text{ and } L_1(x_1) = 1. \]

If \( f(x) \) is any function then

\[ P(x) = f(x_0) L_0(x) + f(x_1) L_1(x) \]

is a linear function which matches \( f(x) \) at \( x = x_0 \) and \( x = x_1 \).
We can take another approach to finding the linear approximation. Define

\[ L_0(x) = \frac{x - x_1}{x_0 - x_1}, \text{ then } L_0(x_0) = 1 \text{ and } L_0(x_1) = 0. \]

\[ L_1(x) = \frac{x - x_0}{x_1 - x_0}, \text{ then } L_1(x_0) = 0 \text{ and } L_1(x_1) = 1. \]

If \( f(x) \) is any function then

\[ P(x) = f(x_0) L_0(x) + f(x_1) L_1(x) \]

is a linear function which matches \( f(x) \) at \( x = x_0 \) and \( x = x_1 \).

We want to extend this idea to higher order polynomials.
Construction of Polynomials

Challenge: given a function $f(x)$, construct a polynomial $P(x)$ of degree at most $n$ which matches $f(x)$ at $n + 1$ distinct points.

$\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\}$
Construction of Polynomials

Challenge: given a function $f(x)$, construct a polynomial $P(x)$ of degree at most $n$ which matches $f(x)$ at $n + 1$ distinct points.

$$\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\}$$

Strategy: construct $n + 1$ polynomials $L_{n,k}(x)$ of degree $n$ with the property that

$$L_{n,k}(x_i) = \begin{cases} 0 & \text{if } k \neq i, \\ 1 & \text{if } k = i \end{cases}$$

for $k = 0, 1, 2, \ldots, n$. 
Lagrange Basis Polynomials

Definition
Suppose \( \{x_0, x_1, \ldots, x_n\} \) is a set of \( n + 1 \) distinct points. A Lagrange Basis Polynomial is a polynomial of degree \( n \) having the form

\[
L_{n,k}(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}.
\]
Lagrange Basis Polynomials

Definition
Suppose \( \{x_0, x_1, \ldots, x_n\} \) is a set of \( n + 1 \) distinct points. A Lagrange Basis Polynomial is a polynomial of degree \( n \) having the form

\[
L_{n,k}(x) = \prod_{i=0, i \neq k}^{n} \frac{x - x_i}{x_k - x_i}.
\]

Remarks:
- We will let \( k = 0, 1, 2, \ldots, n \).
- \( L_{n,k}(x_i) = 1 \) for \( i = k \).
- \( L_{n,k}(x_i) = 0 \) for \( i \neq k \).
Example

If $x_i = i$ for $i = 0, 1, 2$ then we can create three different Lagrange Basis Polynomials.

\[
L_{2,0}(x) =
\]

\[
L_{2,1}(x) =
\]

\[
L_{2,2}(x) =
\]
Example

If $x_i = i$ for $i = 0, 1, 2$ then we can create three different Lagrange Basis Polynomials.

$L_{2,0}(x) = \frac{x-1}{0-1} \cdot \frac{x-2}{0-2} = \frac{1}{2}(x-1)(x-2)$

$L_{2,1}(x) =$

$L_{2,2}(x) =$
Example

If $x_i = i$ for $i = 0, 1, 2$ then we can create three different Lagrange Basis Polynomials.

$L_{2,0}(x) = \frac{x - 1}{0 - 1} \cdot \frac{x - 2}{0 - 2} = \frac{1}{2} (x - 1)(x - 2)$

$L_{2,1}(x) = \frac{x - 0}{1 - 0} \cdot \frac{x - 2}{1 - 2} = x(2 - x)$

$L_{2,2}(x) =$
Example

If \( x_i = i \) for \( i = 0, 1, 2 \) then we can create three different Lagrange Basis Polynomials.

\[
L_{2,0}(x) = \frac{x - 1}{0 - 1} \cdot \frac{x - 2}{0 - 2} = \frac{1}{2}(x - 1)(x - 2)
\]

\[
L_{2,1}(x) = \frac{x - 0}{1 - 0} \cdot \frac{x - 2}{1 - 2} = x(2 - x)
\]

\[
L_{2,2}(x) = \frac{x - 0}{2 - 0} \cdot \frac{x - 1}{2 - 1} = \frac{1}{2}x(x - 1)
\]
Graph

\[ L_{2,0}(x) = \frac{1}{2}(x - 1)(x - 2) \]
\[ L_{2,1}(x) = x(2 - x) \]
\[ L_{2,2}(x) = \frac{1}{2}x(x - 1) \]
Lagrange Interpolating Polynomials

Definition
Let \( \{x_0, x_1, \ldots, x_n\} \) be a set of \( n + 1 \) distinct points at which the function \( f(x) \) is defined. The (unique) Lagrange Interpolating Polynomial of \( f(x) \) of degree \( n \) is the polynomial having the form

\[
P(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x).
\]

Properties:
- \( L_{n,k}(x_i) = 0 \) for all \( i \neq k \).
- \( L_{n,k}(x_k) = 1 \).
- \( P(x_k) = f(x_k) \) for \( k = 0, 1, \ldots, n \).
Example

If $n = 9$ and the nodes are $\{0, 1, \ldots, 9\}$ then $L_{9,4}(x)$ has a graph resembling the following.
Let $f(x) = \sin x$, $n = 3$, and $x_k = k\pi/3$ for $k = 0, 1, 2, 3$. The four Lagrange Basis Polynomials are listed below.

$$L_{3,0}(x) = \frac{x - \frac{\pi}{3}}{0 - \frac{\pi}{3}} \cdot \frac{x - \frac{2\pi}{3}}{0 - \frac{2\pi}{3}} \cdot \frac{x - \pi}{0 - \pi} = \frac{1}{2\pi^3} (\pi - 3x)(2\pi - 3x)(\pi - x)$$

$$L_{3,1}(x) = \frac{9}{2\pi^3} x(2\pi - 3x)(\pi - x)$$

$$L_{3,2}(x) = -\frac{9}{2\pi^3} x(\pi - 3x)(\pi - x)$$

$$L_{3,3}(x) = \frac{1}{2\pi^3} x(\pi - 3x)(2\pi - 3x)$$
The Lagrange Interpolating Polynomial of degree 3 for $f(x) = \sin x$ and $x_k = k\pi/3$ for $k = 0, 1, 2, 3$ is

$$P(x) = (\sin 0) L_{3,0}(x) + \left(\sin \frac{\pi}{3}\right) L_{3,1}(x) + \left(\sin \frac{2\pi}{3}\right) L_{3,2}(x)$$

$$+ (\sin \pi) L_{3,3}(x)$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{9}{2\pi^3} x(2\pi - 3x)(\pi - x) - \frac{\sqrt{3}}{2} \cdot \frac{9}{2\pi^3} x(\pi - 3x)(\pi - x)$$

$$= \frac{9\sqrt{3}}{4\pi^2} x(\pi - x)$$
The graphs of $P(x)$, $f(x)$, and the absolute error are shown below.
Lagrange Polynomial with Error Term

Theorem
Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and suppose $f \in C^{n+1}[a, b]$. Then for each $x \in [a, b]$ there exists a number $z(x) \in (a, b)$ for which

$$f(x) = P(x) + \frac{f^{(n+1)}(z(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n),$$

where $P(x)$ is the Lagrange Interpolating Polynomial.
Compare the error terms of the Lagrange polynomial and the Taylor polynomial.

Taylor: \[ \frac{f^{(n+1)}(z(x))}{(n+1)!} (x - x_0)^{n+1} \]

Lagrange: \[ \frac{f^{(n+1)}(z(x))}{(n+1)!} (x - x_0)(x - x_1) \cdots (x - x_n) \]

Lagrange polynomials form the basis of many numerical approximations to derivatives and integrals, and thus the error term is important to understanding the errors present in those approximations.
Earlier we found the Lagrange interpolating polynomial of degree 3 for $f(x) = \sin x$ using nodes $x_k = k\pi/3$ for $k = 0, 1, 2, 3$ was

$$P(x) = \frac{9\sqrt{3}}{4\pi^2}x(\pi - x).$$

What is an error bound for this approximation?
Earlier we found the Lagrange interpolating polynomial of degree 3 for $f(x) = \sin x$ using nodes $x_k = k\pi/3$ for $k = 0, 1, 2, 3$ was

$$P(x) = \frac{9\sqrt{3}}{4\pi^2} x(\pi - x).$$

What is an error bound for this approximation?

$$|R(x)| = \left| \frac{\sin z}{4!} (x - 0) \left(x - \frac{\pi}{3}\right) \left(x - \frac{2\pi}{3}\right) (x - \pi) \right| \leq 0.050108$$
Error Bound vs. Actual Error
Example

Suppose we are preparing a table of values for $\cos x$ on $[0, \pi]$. The entries in the table will have eight accurate decimal places and we will linearly interpolate between adjacent entries to determine intermediate values. What should the spacing between adjacent $x$-values be to preserve the eight-decimal-place accuracy in the interpolation?
\[ |\cos x - P(x)| = \left| \frac{-\cos z}{2} \right| |x - x_j||x - x_{j+1}| \] for some \( 0 \leq z \leq \pi \)

\[ \leq \frac{1}{2} \max_{0 \leq z \leq \pi} |\cos z| \cdot \max_{x_j \leq x \leq x_{j+1}} |x - x_j||x - x_{j+1}| \]

\[ = \frac{1}{2} \max_{j h \leq x \leq (j+1) h} |(x - jh)(x - (j + 1)h)| \]

\[ = \frac{h^2}{8} \]

Thus if \( h^2/8 < 10^{-8} \) then \( h < 2\sqrt{2} \times 10^{-4} \approx 0.000282 \).
Homework

▶ Read Section 3.1
▶ Exercises: 1, 3, 5a, 7a, 13, 17