Richardson’s Extrapolation
MATH 375 Numerical Analysis

J. Robert Buchanan

Department of Mathematics

Spring 2019
Objectives

Recall the centered-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f^{(3)}(z)$$

truncation error

In today’s lesson we will learn to create higher-accuracy approximations while using lower-order formulas.

The technique, known as **extrapolation** can be used whenever the truncation error has a predictable form (as above) and depends on a parameter such as $h$, the step size.
General Setting

Suppose that $N_1(h)$ is a formula which approximates a quantity $M$.

$$N_1(h) \approx M$$
Suppose that $N_1(h)$ is a formula which approximates a quantity $M$.

$$N_1(h) \approx M$$

Imagine the truncation error of this approximation can be written as

$$M - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

for some unknown constants $K_1, K_2, K_3, \ldots$. 
General Setting

Suppose that \(N_1(h)\) is a formula which approximates a quantity \(M\).

\[ N_1(h) \approx M \]

Imagine the truncation error of this approximation can be written as

\[ M - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots \]

for some unknown constants \(K_1, K_2, K_3, \ldots\).

Note that

\[
\begin{align*}
M - N_1(0.1) &= K_1(0.1) + K_2(0.1)^2 + \cdots \approx (0.1)K_1 \\
M - N_1(0.01) &= K_1(0.01) + K_2(0.01)^2 + \cdots \approx (0.01)K_1
\end{align*}
\]

and in general \(M - N_1(h) \approx K_1 h\).
Order of the Truncation Error

**Question:** Since the truncation error is $O(h)$, can we combine several $O(h)$ approximations to create an $O(h^n)$ approximation where $n \geq 2$?
Order of the Truncation Error

**Question:** Since the truncation error is $O(h)$, can we combine several $O(h)$ approximations to create an $O(h^n)$ approximation where $n \geq 2$?

\[
M = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots
\]

\[
M = N_1 \left( \frac{h}{2} \right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + \cdots
\]
Order of the Truncation Error

**Question:** Since the truncation error is $O(h)$, can we combine several $O(h)$ approximations to create an $O(h^n)$ approximation where $n \geq 2$?

\[
M = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots
\]

\[
M = N_1 \left( \frac{h}{2} \right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + \cdots
\]

Multiply the 2nd equation by 2 and subtract the 1st equation.

\[
M = 2N_1 \left( \frac{h}{2} \right) - N_1(h) + K_2 \left[ \frac{h^2}{2} - h^2 \right] + K_3 \left[ \frac{h^3}{4} - h^3 \right] + \cdots
\]

\[
= N_1 \left( \frac{h}{2} \right) + \left[ N_1 \left( \frac{h}{2} \right) - N_1(h) \right] - \frac{K_2}{2} h^2 - \frac{3K_3}{4} h^3 - \cdots
\]

**Note:** the $O(h)$ truncation terms have vanished.
A Second Approximation

Define $N_2(h) = N_1 \left( \frac{h}{2} \right) + \left[ N_1 \left( \frac{h}{2} \right) - N_1(h) \right]$ and then we have

$$M = N_2(h) - \frac{K_2}{2} h^2 - \frac{3K_3}{4} h^3 - \ldots$$

which has $O(h^2)$ truncation error.

**Note:** we have combined multiple $O(h)$ approximations to generate an $O(h^2)$ approximation.
Example

Recall the 2-point forward-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{h}(f(x_0 + h) - f(x_0)) - \frac{f''(z)}{2}h$$

This is an $O(h)$ truncation error.
Example

Recall the 2-point forward-difference formula for \( f'(x_0) \):

\[
f'(x_0) = \frac{1}{h} (f(x_0 + h) - f(x_0)) - \frac{f''(z)}{2} h
\]

This is an \( O(h) \) truncation error.

Let \( f(x) = x \sin x \) then we have:

<table>
<thead>
<tr>
<th>( h )</th>
<th>( f'(1) \approx N_1(h) )</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>1.38857</td>
<td>0.00679782</td>
</tr>
<tr>
<td>0.05</td>
<td>1.38647</td>
<td>0.00469475</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

\[
N_2(0.1) = N_1(0.05) + (N_1(0.05) - N_1(0.1)) = 1.38436
\]
Example

Recall the 2-point forward-difference formula for \( f'(x_0) \):

\[
f'(x_0) = \frac{1}{h} (f(x_0 + h) - f(x_0)) - \frac{f''(z)}{2} h
\]

This is an \( O(h) \) truncation error.

Let \( f(x) = x \sin x \) then we have:

<table>
<thead>
<tr>
<th>( h )</th>
<th>( f'(1) \approx N_1(h) )</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>1.38857</td>
<td>0.00679782</td>
</tr>
<tr>
<td>0.05</td>
<td>1.38647</td>
<td>0.00469475</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

\[
N_2(0.1) = N_1(0.05) + (N_1(0.05) - N_1(0.1)) = 1.38436
\]

Note that \( |N_2(0.1) - f'(1)| \approx 0.00259168 \).
Use Richardson’s Extrapolation and the 2-point forward difference formula for $f'(x_0)$ to develop an $O(h^2)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$. 

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + \left( N_1(0.05) - N_1(0.1) \right) = -5.30499$$

Note that $|N_2(0.1) - f'(2)| \approx 0.00320877$. 

Example
Example

Use Richardson’s Extrapolation and the 2-point forward difference formula for $f'(x_0)$ to develop an $O(h^2)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$f'(2) \approx N_1(h)$</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>-5.61784</td>
<td>0.316063</td>
</tr>
<tr>
<td>0.05</td>
<td>-5.46141</td>
<td>0.159636</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + (N_1(0.05) - N_1(0.1)) = -5.30499$$
Example

Use Richardson’s Extrapolation and the 2-point forward difference formula for \( f'(x_0) \) to develop an \( O(h^2) \) approximation to \( f'(2) \) where \( f(x) = x^2 \cos x \) using \( h = 0.1 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( f'(2) \approx N_1(h) )</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>-5.61784</td>
<td>0.316063</td>
</tr>
<tr>
<td>0.05</td>
<td>-5.46141</td>
<td>0.159636</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

\[
N_2(0.1) = N_1(0.05) + (N_1(0.05) - N_1(0.1)) = -5.30499
\]

Note that \( |N_2(0.1) - f'(2)| \approx 0.00320877 \).
Improving Richardson’s Extrapolation

**Remark:** If the truncation error contains only even powers of $h$, the extrapolation is more effective.

Suppose

$$M = N_1(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

$$M = N_1 \left( \frac{h}{2} \right) + K_1 \frac{h^2}{4} + K_2 \frac{h^4}{16} + K_3 \frac{h^6}{64} + \cdots$$
**Remark:** If the truncation error contains only even powers of $h$, the extrapolation is more effective.

Suppose

\[
M = N_1(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots
\]

\[
M = N_1 \left( \frac{h}{2} \right) + K_1 \frac{h^2}{4} + K_2 \frac{h^4}{16} + K_3 \frac{h^6}{64} + \cdots
\]

Multiply the 2nd equation by 4 and subtract the 1st equation.

\[
3M = \left[ 4N_1 \left( \frac{h}{2} \right) - N_1(h) \right] + K_2 \left[ \frac{h^4}{4} - h^4 \right] + K_3 \left[ \frac{h^6}{16} - h^6 \right] + \cdots
\]
If we multiply the previous equation by $1/3$ we obtain

\[ M = \frac{1}{3} \left[ 4N_1 \left( \frac{h}{2} \right) - N_1(h) \right] + \frac{K_2}{3} \left[ \frac{h^4}{4} - h^4 \right] + \frac{K_3}{3} \left[ \frac{h^6}{16} - h^6 \right] + \cdots \]
$O(h^4)$ Truncation Error

If we multiply the previous equation by $1/3$ we obtain

$$M = \frac{1}{3} \left[ 4N_1 \left( \frac{h}{2} \right) - N_1(h) \right] + \frac{K_2}{3} \left[ \frac{h^4}{4} - h^4 \right] + \frac{K_3}{3} \left[ \frac{h^6}{16} - h^6 \right] + \cdots$$

Define

$$N_2(h) = \frac{1}{3} \left[ 4N_1 \left( \frac{h}{2} \right) - N_1(h) \right] = N_1 \left( \frac{h}{2} \right) + \frac{1}{3} \left[ N_1 \left( \frac{h}{2} \right) - N_1(h) \right].$$
$O(h^4)$ Truncation Error

If we multiply the previous equation by $1/3$ we obtain

$$M = \frac{1}{3} \left[ 4N_1 \left( \frac{h}{2} \right) - N_1(h) \right] + \frac{K_2}{3} \left[ \frac{h^4}{4} - h^4 \right] + \frac{K_3}{3} \left[ \frac{h^6}{16} - h^6 \right] + \cdots$$

Define

$$N_2(h) = \frac{1}{3} \left[ 4N_1 \left( \frac{h}{2} \right) - N_1(h) \right] = N_1 \left( \frac{h}{2} \right) + \frac{1}{3} \left[ N_1 \left( \frac{h}{2} \right) - N_1(h) \right].$$

This is an approximation formula with truncation error $O(h^4)$.

$$M = N_2(h) - \frac{K_2}{4} h^4 - \frac{5K_3}{16} h^6 + \cdots$$
Example

Recall the 3-point centered-difference formula for \( f'(x_0) \):

\[
f'(x_0) = \frac{1}{2h} (f(x_0 + h) - f(x_0 - h)) - \frac{f'''(z)}{6} h^2
\]

This is an \( O(h^2) \) truncation error.
**Example**

Recall the 3-point centered-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h}(f(x_0 + h) - f(x_0 - h)) - \frac{f'''(z)}{6}h^2$$

This is an $O(h^2)$ truncation error.

Let $f(x) = x \sin x$ then we have:

<table>
<thead>
<tr>
<th>$h$</th>
<th>$f'(1) \approx N_1(h)$</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>1.37667</td>
<td>0.0051039</td>
</tr>
<tr>
<td>0.05</td>
<td>1.38050</td>
<td>0.0012767</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + \frac{1}{3}(N_1(0.05) - N_1(0.1)) = 1.38177$$
Example

Recall the 3-point centered-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h}(f(x_0 + h) - f(x_0 - h)) - \frac{f'''(z)}{6} h^2$$

This is an $O(h^2)$ truncation error.

Let $f(x) = x \sin x$ then we have:

<table>
<thead>
<tr>
<th>$h$</th>
<th>$f'(1) \approx N_1(h)$</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>1.37667</td>
<td>0.0051039</td>
</tr>
<tr>
<td>0.05</td>
<td>1.38050</td>
<td>0.0012767</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + \frac{1}{3}(N_1(0.05) - N_1(0.1)) = 1.38177$$

Note that $|N_2(0.1) - f'(1)| \approx 9.88697 \times 10^{-7}$. 
Example

Use Richardson’s Extrapolation and the 3-point centered-difference formula for $f'(x_0)$ to develop an $O(h^4)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$. 

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + \frac{1}{3} (N_1(0.05) - N_1(0.1)) = -5.30178$$

Note that $|N_2(0.1) - f'(2)| \approx 1.29563 \times 10^{-6}$. 
Example

Use Richardson’s Extrapolation and the 3-point centered-difference formula for $f'(x_0)$ to develop an $O(h^4)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$f'(2) \approx N_1(h)$</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>$-5.29648$</td>
<td>0.00529713</td>
</tr>
<tr>
<td>0.05</td>
<td>$-5.30045$</td>
<td>0.00132331</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + \frac{1}{3}(N_1(0.05) - N_1(0.1)) = -5.30178$$
Example

Use Richardson’s Extrapolation and the 3-point centered-difference formula for $f'(x_0)$ to develop an $O(h^4)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$f'(2) \approx N_1(h)$</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>-5.29648</td>
<td>0.00529713</td>
</tr>
<tr>
<td>0.05</td>
<td>-5.30045</td>
<td>0.00132331</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + \frac{1}{3}(N_1(0.05) - N_1(0.1)) = -5.30178$$

Note that $|N_2(0.1) - f'(2)| \approx 1.29563 \times 10^{-6}$. 
Return to the $O(h^4)$ Formula

Recall:

$$M = N_2(h) - \frac{K_2}{4} h^4 - \frac{5K_3}{16} h^6 + \cdots$$
Return to the $O(h^4)$ Formula

Recall:

$$M = N_2(h) - \frac{K_2}{4} h^4 - \frac{5K_3}{16} h^6 + \cdots$$

Replace $h$ by $h/2$:

$$M = N_2 \left( \frac{h}{2} \right) - \frac{K_2}{64} h^4 - \frac{5K_3}{1024} h^6 + \cdots$$

which is also has an $O(h^4)$ truncation error.
Return to the \( O(h^4) \) Formula

Recall:

\[
M = N_2(h) - \frac{K_2}{4} h^4 - \frac{5K_3}{16} h^6 + \cdots
\]

Replace \( h \) by \( h/2 \):

\[
M = N_2 \left( \frac{h}{2} \right) - \frac{K_2}{64} h^4 - \frac{5K_3}{1024} h^6 + \cdots
\]

which is also has an \( O(h^4) \) truncation error.

Multiply the 2nd equation by 16 and subtract the first equation from it.

\[
15M = \left[ 16N_2 \left( \frac{h}{2} \right) - N_2(h) \right] + \frac{15K_3}{64} h^6 + \cdots
\]
$O(h^6)$ Truncation Error

Multiplying both sides of the last equation by $1/15$ yields:

$$M = \frac{1}{15} \left[ 16N_2 \left( \frac{h}{2} \right) - N_2(h) \right] + \frac{K_3}{64} h^6 + \cdots$$
Multiplying both sides of the last equation by $1/15$ yields:

$$M = \frac{1}{15} \left[ 16N_2 \left( \frac{h}{2} \right) - N_2(h) \right] + \frac{K_3}{64}h^6 + \cdots$$

We can define

$$N_3(h) = N_2 \left( \frac{h}{2} \right) + \frac{1}{15} \left[ N_2 \left( \frac{h}{2} \right) - N_2(h) \right].$$

This approximation formula has an $O(h^6)$ truncation error.
Example

We will use two approximations to $f'(1)$ where $f(x) = x \sin x$ with $O(h^4)$ truncation errors to develop an approximation with $O(h^6)$ truncation error.
Example

We will use two approximations to $f'(1)$ where $f(x) = x \sin x$ with $O(h^4)$ truncation errors to develop an approximation with $O(h^6)$ truncation error.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$f'(1) \approx N_2(h)$</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>1.38177</td>
<td>$9.88697 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.05</td>
<td>1.38177</td>
<td>$6.18122 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

$$N_3(0.1) = N_2(0.05) + \frac{1}{15}(N_2(0.05) - N_2(0.1)) = 1.38177$$
Example

We will use two approximations to \( f'(1) \) where \( f(x) = x \sin x \) with \( O(h^4) \) truncation errors to develop an approximation with \( O(h^6) \) truncation error.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( f'(1) \approx N_2(h) )</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>1.38177</td>
<td>( 9.88697 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.05</td>
<td>1.38177</td>
<td>( 6.18122 \times 10^{-8} )</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

\[
N_3(0.1) = N_2(0.05) + \frac{1}{15} (N_2(0.05) - N_2(0.1)) = 1.38177
\]

Note that \( |N_3(0.1) - f'(1)| \approx 1.99358 \times 10^{-11} \).
Example

Use Richardson’s Extrapolation and the 3-point centered-difference formula for $f'(x_0)$ to develop an $O(h^6)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$. 

Applying the extrapolation formula gives us another approximation: 

$N_3(0.1) = N_2(0.05) + \frac{1}{15}(N_2(0.05) - N_2(0.1)) = -5.30178$ 

Note that $|N_3(0.1) - f'(2)| \approx 7.09512 \times 10^{-11}$. 

\[
\begin{align*}
N_2(0.05) & \approx N_2(0.1) \\
N_3(0.1) & \approx N_2(0.05) + \frac{1}{15}(N_2(0.05) - N_2(0.1)) = -5.30178 \\
\end{align*}
\]
Example

Use Richardson’s Extrapolation and the 3-point centered-difference formula for $f'(x_0)$ to develop an $O(h^6)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$f'(2) \approx N_2(h)$</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>-5.30178</td>
<td>$1.29563 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.05</td>
<td>-5.30178</td>
<td>$8.10432 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

$$N_3(0.1) = N_2(0.05) + \frac{1}{15}(N_2(0.05) - N_2(0.1)) = -5.30178$$
Example

Use Richardson’s Extrapolation and the 3-point centered-difference formula for \( f'(x_0) \) to develop an \( O(h^6) \) approximation to \( f'(2) \) where \( f(x) = x^2 \cos x \) using \( h = 0.1 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( f'(2) \approx N_2(h) )</th>
<th>Abs. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>−5.30178</td>
<td>( 1.29563 \times 10^{-6} )</td>
</tr>
<tr>
<td>0.05</td>
<td>−5.30178</td>
<td>( 8.10432 \times 10^{-8} )</td>
</tr>
</tbody>
</table>

Applying the extrapolation formula gives us another approximation:

\[
N_3(0.1) = N_2(0.05) + \frac{1}{15}(N_2(0.05) - N_2(0.1)) = −5.30178
\]

Note that \( |N_3(0.1) - f'(2)| \approx 7.09512 \times 10^{-11} \).
For $j = 2, 3, \ldots$ the $O(h^{2j})$ truncation error approximation is given by the formula

$$N_j(h) = N_{j-1} \left( \frac{h}{2} \right) + \frac{1}{4^{j-1} - 1} \left[ N_{j-1} \left( \frac{h}{2} \right) - N_{j-1}(h) \right].$$
General Situation

For $j = 2, 3, \ldots$ the $O(h^{2^j})$ truncation error approximation is given by the formula

$$N_j(h) = N_{j-1} \left( \frac{h}{2} \right) + \frac{1}{4^{j-1} - 1} \left[ N_{j-1} \left( \frac{h}{2} \right) - N_{j-1}(h) \right].$$

For $j = 2, 3, \ldots$ the $O(h^j)$ truncation error approximation is given by the formula

$$N_j(h) = N_{j-1} \left( \frac{h}{2} \right) + \frac{1}{2^{j-1} - 1} \left[ N_{j-1} \left( \frac{h}{2} \right) - N_{j-1}(h) \right].$$
Remark: Richardson’s extrapolation provides a convenient means for developing the 5-point approximations to $f'(x_0)$.

Assuming $f \in C^5[a, b]$ and $x_0 \in (a, b)$ expand $f(x)$ as a degree 4 Taylor polynomial about $x_0$.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$
$$+ \frac{1}{6} f'''(x_0)(x - x_0)^3 + \frac{1}{24} f^{(4)}(x_0)(x - x_0)^4$$
$$+ \frac{1}{120} f^{(5)}(z)(x - x_0)^5$$

where $z$ lies between $x$ and $x_0$. 

Multi-point Differentiation Formulas
Evaluate the Taylor polynomial expansion at $x = x_0 \pm h$.

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 + \frac{1}{120}f^{(5)}(z_1)h^5$$

$$f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 - \frac{1}{120}f^{(5)}(z_2)h^5$$

with $x_0 - h \leq z_2 \leq x_0 \leq z_1 \leq x_0 + h$. 

Now subtract the 2nd equation from the 1st equation.
Evaluate the Taylor polynomial expansion at $x = x_0 \pm h$.

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 + \frac{1}{120}f^{(5)}(z_1)h^5$$

$$f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 - \frac{1}{120}f^{(5)}(z_2)h^5$$

with $x_0 - h \leq z_2 \leq x_0 \leq z_1 \leq x_0 + h$.

Now subtract the 2nd equation from the 1st equation.
Five-Point Formula (2 of 5)

\[ f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3}{3} f''''(x_0) + \frac{h^5}{120} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \]
Five-Point Formula (2 of 5)

\[ f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3}{3} f'''(x_0) + \frac{h^5}{120} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \]

By assuming that \( f \in C^5[a, b] \) we know \( f^{(5)}(x) \) is continuous on \([a, b]\).

Note that

\[ \frac{1}{2} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \]

lies between \( f^{(5)}(z_1) \) and \( f^{(5)}(z_2) \).
Five-Point Formula (2 of 5)

\[ f(x_0+h) - f(x_0-h) = 2hf'(x_0) + \frac{h^3}{3}f'''(x_0) + \frac{h^5}{120} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \]

By assuming that \( f \in C^5[a, b] \) we know \( f^{(5)}(x) \) is continuous on \([a, b]\).

Note that

\[ \frac{1}{2} \left[ f^{(5)}(Z_1) + f^{(5)}(Z_2) \right] \]

lies between \( f^{(5)}(Z_1) \) and \( f^{(5)}(Z_2) \).

According to the Intermediate Value Theorem there exists \( w \) between \( z_1 \) and \( z_2 \) for which

\[
  f^{(5)}(w) = \frac{1}{2} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \\
  2f^{(5)}(w) = f^{(5)}(z_1) + f^{(5)}(z_2)
\]
Thus we may write the Taylor polynomial difference as

\[ f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3}{3}f'''(x_0) + \frac{h^5}{60}f^{(5)}(w) \]

for some \( x_0 - h \leq w \leq x_0 + h \).

Solve this equation for \( f'(x_0) \).
Thus we may write the Taylor polynomial difference as

\[ f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3}{3}f'''(x_0) + \frac{h^5}{60}f^{(5)}(w) \]

for some \( x_0 - h \leq w \leq x_0 + h \).

Solve this equation for \( f'(x_0) \).

\[ f'(x_0) = \frac{1}{2h} \left[ f(x_0 + h) - f(x_0 - h) \right] - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(w) \]

Now apply the Richardson’s extrapolation technique to this approximation.
Five-Point Formula (4 of 5)

\[ f'(x_0) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f'''(x_0) - \frac{h^4}{120} f^{(5)}(w) \]

If we replace \( h \) by \( 2h \) we get

\[ f'(x_0) = \frac{1}{4h} [f(x_0 + 2h) - f(x_0 - 2h)] - \frac{4h^2}{6} f'''(x_0) - \frac{16h^4}{120} f^{(5)}(\tilde{w}) \]

where \( \tilde{w} \) lies between \( x_0 - 2h \) and \( x_0 + 2h \).
Five-Point Formula (4 of 5)

\[ f'(x_0) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f'''(x_0) - \frac{h^4}{120} f^{(5)}(w) \]

If we replace \( h \) by \( 2h \) we get

\[ f'(x_0) = \frac{1}{4h} [f(x_0 + 2h) - f(x_0 - 2h)] - \frac{4h^2}{6} f'''(x_0) - \frac{16h^4}{120} f^{(5)}(\tilde{w}) \]

where \( \tilde{w} \) lies between \( x_0 - 2h \) and \( x_0 + 2h \).

Multiply the 1st equation by 4:

\[ 4f'(x_0) = \frac{4}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{4h^2}{6} f'''(x_0) - \frac{4h^4}{120} f^{(5)}(w) \]

and subtract the 2nd equation.
Five-Point Formula (5 of 5)

\[ 3f'(x_0) = \frac{2}{h} [f(x_0 + h) - f(x_0 - h)] - \frac{1}{4h} [f(x_0 + 2h) - f(x_0 - 2h)] \]
\[ \quad - \frac{4h^4}{120} f^{(5)}(w) + \frac{16h^4}{120} f^{(5)}(\tilde{w}) \]

\[ f'(x_0) = \frac{2}{3h} [f(x_0 + h) - f(x_0 - h)] - \frac{1}{12h} [f(x_0 + 2h) - f(x_0 - 2h)] \]
\[ \quad - \frac{h^4}{90} f^{(5)}(w) + \frac{2h^4}{45} f^{(5)}(\tilde{w}) \]
\[ = \frac{1}{12h} [f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)] \]
\[ \quad - \frac{h^4}{30} f^{(5)}(\hat{w}) \]

The form of the truncation error has not been justified.
Homework

- Read Section 4.2.
- Exercises: 1a, 7, 9