Absolute Value and the Real Line
MATH 464/506, Real Analysis

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Definition
The **absolute value** of a real number \( a \), denoted by \( |a| \), is defined by

\[
|a| = \begin{cases} 
  a & \text{if } a > 0, \\
  0 & \text{if } a = 0, \\
  -a & \text{if } a < 0.
\end{cases}
\]

Theorem
1. \(|ab| = |a||b|\) for all \( a, b \in \mathbb{R} \).
2. \(|a|^2 = a^2\) for all \( a \in \mathbb{R} \).
3. If \( c \geq 0 \), then \(|a| \leq c\) if and only if \(-c \leq a \leq c\).
4. \(-|a| \leq a \leq |a|\) for all \( a \in \mathbb{R} \).
**Absolute Value**

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4. $-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.

**Proof.**
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Proof.
Triangle Inequality

**Theorem**

If $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

**Proof.**

**Corollary**

If $a, b \in \mathbb{R}$, then

1. $||a| - |b|| \leq |a - b|$,  
2. $|a - b| \leq |a| + |b|$.  

If $a_1, a_2, \ldots, a_n$ are any real numbers, then

$$|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|.$$
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\[ |a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|. \]
Remarks:
- Geometrically we may regard $|a|$ as the distance along the number line from 0 to $a$.
- The distance between $a$ and $b$ in $\mathbb{R}$ is $|a - b|$.

Definition
Let $a \in \mathbb{R}$ and $\epsilon > 0$. The $\epsilon$-neighborhood of $a$ is the set

$$ V_\epsilon(a) = \{ x \in \mathbb{R} : |x - a| < \epsilon \}. $$

Remark: $x \in V_\epsilon(a)$ means $x$ satisfies the following equivalent inequalities:

$$ -\epsilon < x - a < \epsilon $$
$$ a - \epsilon < x < a + \epsilon $$
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Theorem

Let \( a \in \mathbb{R} \). If \( x \) belongs to the neighborhood \( V_{\varepsilon}(a) \) for every \( \varepsilon > 0 \), then \( x = a \).

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Homework

- Read Section 2.2.
- Page 34: 1, 2, 14, 15

Boxed problems should be written up separately and submitted for grading at class time on Friday.