Algebraic and Order Properties of $\mathbb{R}$

MATH 464/506, *Real Analysis*

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The real number system \( \mathbb{R} \) is a complete ordered field.
On the set \( \mathbb{R} \) there are two **binary operations**, denoted “+” (addition) and “\( \cdot \)” (multiplication), which behave according the following axioms.

**Additive Axioms:**

(A0) \( \forall x, y \in \mathbb{R}, \exists \) unique element \( x + y \in \mathbb{R} \).

(A1) \( \forall x, y \in \mathbb{R}, x + y = y + x \) (commutativity of +).

(A2) \( \forall x, y, z \in \mathbb{R}, x + (y + z) = (x + y) + z \) (associativity of +).

(A3) \( \exists \) element 0 \( \in \mathbb{R} \) \( \ni \) \( \forall x \in \mathbb{R}, x + 0 = x \) (existence of additive identity element).

(A4) \( \forall x \in \mathbb{R}, \exists u \in \mathbb{R} \) \( \ni x + u = 0 \) (existence of additive inverses).
Multiplicative Axioms:

(M0) \( \forall x, y \in \mathbb{R}, \exists \) unique element \( x \cdot y \in \mathbb{R} \).

(M1) \( \forall x, y \in \mathbb{R}, x \cdot y = y \cdot x \) (commutativity of \( \cdot \)).

(M2) \( \forall x, y, z \in \mathbb{R}, x \cdot (y \cdot z) = (x \cdot y) \cdot z \) (associativity of \( \cdot \)).

(M3) \( \exists \) element \( 1 \in \mathbb{R} \) \( \ni 1 \neq 0 \) and \( \forall x \in \mathbb{R}, x \cdot 1 = x \) (existence of multiplicative identity element).

(M4) \( \forall x \in \mathbb{R} \) \( \exists x \neq 0, \exists u \in \mathbb{R} \) \( \ni x \cdot u = 1 \) (existence of multiplicative inverses).

Distributive Axiom:

(D) \( \forall x, y, z \in \mathbb{R}, x \cdot (y + z) = (x \cdot y) + (x \cdot z) \).
Properties of Identity Elements

Theorem

1. If $z$ and $a$ are elements of $\mathbb{R}$ with $z + a = a$, then $z = 0$.
2. If $u$ and $b \neq 0$ are elements of $\mathbb{R}$ with $u \cdot b = b$, then $u = 1$.
3. If $a \in \mathbb{R}$, then $a \cdot 0 = 0$. 

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Properties of Identity Elements

Theorem

1. If \( z \) and \( a \) are elements of \( \mathbb{R} \) with \( z + a = a \), then \( z = 0 \).
2. If \( u \) and \( b \neq 0 \) are elements of \( \mathbb{R} \) with \( u \cdot b = b \), then \( u = 1 \).
3. If \( a \in \mathbb{R} \), then \( a \cdot 0 = 0 \).

Proof.
Further Properties of Multiplication

Theorem

(Uniqueness of Reciprocals) If $a \neq 0$ and $b \in \mathbb{R}$ are such that $a \cdot b = 1$, then $b = 1/a$.

(Zero Factor Property) If $a \cdot b = 0$, then either $a = 0$ or $b = 0$. 
Further Properties of Multiplication

**Theorem**

*Uniqueness of Reciprocals* If \( a \neq 0 \) and \( b \in \mathbb{R} \) are such that \( a \cdot b = 1 \), then \( b = 1/a \).

*Zero Factor Property* If \( a \cdot b = 0 \), then either \( a = 0 \) or \( b = 0 \).

**Proof.**
Further Operations

Definition

(Subtraction) \( \forall x, y \in \mathbb{R}, \text{ define } x - y = x + (-y). \)

(Division) \( \forall x, y \in \mathbb{R}, \text{ if } y \neq 0 \text{ define } x \div y = x \cdot (1/y). \)

(Exponentiation) \( \forall x \in \mathbb{R} \text{ and } n \in \mathbb{N}, x^n = \underbrace{x \cdot x \cdots x}_{n \text{ factors}}. \)

\( \forall x \in \mathbb{R} \text{ with } x \neq 0 \text{ and } n \in \mathbb{N}, x^{-n} = (1/x)^n. \)

\( \forall x \in \mathbb{R} \text{ with } x \neq 0, x^0 = 1. \)
Theorem

*There does not exist a rational number \( r \) such that \( r^2 = 2 \).*
Theorem

There does not exist a rational number \( r \) such that \( r^2 = 2 \).

Proof.
Order Axioms:

There is a nonempty subset $P$ of $\mathbb{R}$, called the set of positive real numbers, that possess the following properties:

1. $\forall x, y \in P, \ x + y \in P$ (closure under $+$).
2. $\forall x, y \in P, \ x \cdot y \in P$ (closure under $\cdot$).
3. $\forall x \in \mathbb{R}$, one and only one of the following holds (trichotomy):
   
   $$x \in P, \quad -x \in P, \quad x = 0.$$
If \( a \in \mathbb{P} \) then \( a > 0 \) and we say \( a \) is **positive** or **strictly positive**.

If \( a \in \mathbb{P} \cup \{0\} \) then \( a \geq 0 \) and we say \( a \) is **nonnegative**.

If \( -a \in \mathbb{P} \) then \( a < 0 \) and we say \( a \) is **negative** or **strictly negative**.

If \( -a \in \mathbb{P} \cup \{0\} \) then \( a \leq 0 \) and we say \( a \) is **nonpositive**.
Inequality Between Real Numbers

Definition

Let $a, b \in \mathbb{R}$.

1. If $a - b \in \mathbb{P}$, then we write $a > b$ or $b < a$.
2. If $a - b \in \mathbb{P} \cup \{0\}$, then we write $a \geq b$ or $b \leq a$. 

Title:

Inequality Between Real Numbers

Subheading:

Definition

Text:

Let $a, b \in \mathbb{R}$.

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Theorem

Let \( a, b, c \in \mathbb{R} \).

1. If \( a > b \) and \( b > c \), then \( a > c \).
2. If \( a > b \), then \( a + c > b + c \).
3. If \( a > b \) and \( c > 0 \), then \( ca > cb \).
4. If \( a > b \) and \( c < 0 \), then \( ca < cb \).
Inequality Results

**Theorem**

Let $a, b, c \in \mathbb{R}$.

1. If $a > b$ and $b > c$, then $a > c$.
2. If $a > b$, then $a + c > b + c$.
3. If $a > b$ and $c > 0$, then $ca > cb$.
4. If $a > b$ and $c < 0$, then $ca < cb$.

**Proof.**

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Algebraic and Order Properties of $\mathbb{R}$
Theorem

1. If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.
2. $1 > 0$
3. If $n \in \mathbb{N}$, then $n > 0$. 
Theorem

1. If \( a \in \mathbb{R} \text{ and } a \neq 0 \), then \( a^2 > 0 \).
2. \( 1 > 0 \)
3. If \( n \in \mathbb{N} \), then \( n > 0 \).

Proof.
Largest and Smallest Elements

Theorem

- \( \mathbb{P} \) has no largest and no smallest elements.
- \( \mathbb{R} \) has no smallest positive element and no largest negative element.
- \( \mathbb{P} \) (and consequently \( \mathbb{R} \) itself) is an infinite set.
Theorem

\[ \mathbb{P} \text{ has no largest and no smallest elements.} \]
\[ \mathbb{R} \text{ has no smallest positive element and no largest negative element.} \]
\[ \mathbb{P} \text{ (and consequently } \mathbb{R} \text{ itself) is an infinite set.} \]

Theorem

If \( a \in \mathbb{R} \) is such that \( 0 \leq a < \epsilon \) for every \( \epsilon > 0 \), then \( a = 0 \).
Largest and Smallest Elements

Theorem

- \( \mathbb{P} \) has no largest and no smallest elements.
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Theorem

- If \( a \in \mathbb{R} \) is such that \( 0 \leq a < \varepsilon \) for every \( \varepsilon > 0 \), then \( a = 0 \).

Proof.
Theorem

If $ab > 0$, then either

1. $a > 0$ and $b > 0$, or
2. $a < 0$ and $b < 0$. 
**Theorem**

*If* \( ab > 0 \), *then either*

1. \( a > 0 \) *and* \( b > 0 \), *or*
2. \( a < 0 \) *and* \( b < 0 \).

**Proof.**
Theorem

If $ab > 0$, then either
1. $a > 0$ and $b > 0$, or
2. $a < 0$ and $b < 0$.

Proof.

Corollary

If $ab < 0$, then either
1. $a > 0$ and $b < 0$, or
2. $a < 0$ and $b > 0$. 
Inequalities

Definition
If $a$ and $b$ are positive real numbers, then

- the **arithmetic mean** is $\frac{1}{2}(a + b)$,
- the **geometric mean** is $\sqrt{ab}$. 

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Algebraic and Order Properties of $\mathbb{R}$
Inequalities

**Definition**

If $a$ and $b$ are positive real numbers, then

- the **arithmetic mean** is $\frac{1}{2}(a + b)$,
- the **geometric mean** is $\sqrt{ab}$.

**Theorem**

*(Arithmetic-Geometric Mean Inequality)* If $a$ and $b$ are positive real numbers, then

$$\sqrt{ab} \leq \frac{1}{2}(a + b),$$

with equality holding if and only if $a = b$. 
Inequalities

**Definition**

If \( a \) and \( b \) are positive real numbers, then

- the **arithmetic mean** is \( \frac{1}{2}(a + b) \),
- the **geometric mean** is \( \sqrt{ab} \).

**Theorem**

*(Arithmetic-Geometric Mean Inequality)* If \( a \) and \( b \) are positive real numbers, then

\[
\sqrt{ab} \leq \frac{1}{2}(a + b),
\]

with equality holding if and only if \( a = b \).

**Proof.**
Bernoulli’s Inequality

Theorem

If $x > -1$, then

$$(1 + x)^n \geq 1 + nx$$

for all $n \in \mathbb{N}$. 
Bernoulli’s Inequality

Theorem

If $x > -1$, then

$$(1 + x)^n \geq 1 + nx$$

for all $n \in \mathbb{N}$. 

Proof.
Homework

Read Section 2.1.

Pages 29-30: 1, 3, 4, 6, 9, 13, 18, 19

Boxed problems should be written up separately and handed in for grading at class time on Friday.