The Completeness Property of $\mathbb{R}$
MATH 464/506, Real Analysis

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Summer 2007
We began this Chapter by stating that $\mathbb{R}$ is a complete ordered field. So far we have discussed the algebraic field and ordering properties of $\mathbb{R}$. The algebraic and ordering properties of $\mathbb{R}$ are also shared by $\mathbb{Q}$. The completeness property is not.
Upper and Lower Bounds

Definition

Let $S$ be a nonempty subset of $\mathbb{R}$.

1. The set $S$ is said to be **bounded above** if there exists a number $u \in \mathbb{R}$ such that $s \leq u$ for all $s \in S$. Each such number $u$ is called an **upper bound** of $S$.

2. The set $S$ is said to be **bounded below** if there exists a number $w \in \mathbb{R}$ such that $s \geq w$ for all $s \in S$. Each such number $w$ is called an **lower bound** of $S$.

3. A set is said to be **bounded** if it is both bounded above and below. A set is said to be **unbounded** if it is not bounded.

Sketch
Definition

Let $S$ be a nonempty subset of $\mathbb{R}$.

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Sketch
Suprema and Infima

Definition

Let $S$ be a nonempty subset of $\mathbb{R}$.

1. If $S$ is bounded above, then a number $u$ is said to be a supremaum (or a least upper bound) of $S$ if it satisfies the conditions:
   - $u$ is an upper bound of $S$, and
   - if $v$ is any upper bound of $S$, then $u \leq v$.

2. If $S$ is bounded below, then a number $w$ is said to be an infimum (or a greatest lower bound) of $S$ if it satisfies the conditions:
   - $w$ is an lower bound of $S$, and
   - if $t$ is any lower bound of $S$, then $t \leq w$.

Notation: $u = \sup S$ and $w = \inf S$. 
Theorem

- A set cannot have more than one infimum.
- A set cannot have more than one supremum.

Lemma

A number $u$ is the supremum of a nonempty subset $S$ of $\mathbb{R}$ if and only if $u$ satisfies the conditions:

1. $s \leq u$ for all $s \in S$,
2. if $v < u$, then there exists $s' \in S$ such that $v < s'$. 

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The Completeness Property of $\mathbb{R}$
Theorem

- A set cannot have more than one infimum.
- A set cannot have more than one supremum.

Lemma

A number \( u \) is the supremum of a nonempty subset \( S \) of \( \mathbb{R} \) if and only if \( u \) satisfies the conditions:

1. \( s \leq u \) for all \( s \in S \),
2. if \( v < u \), then there exists \( s' \in S \) such that \( v < s' \).
Lemma

An upper bound $u$ of a nonempty set $S$ in $\mathbb{R}$ is the supremum of $S$ if and only if for every $\epsilon > 0$ there exists an $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$.

Proof.
Lemma

An upper bound $u$ of a nonempty set $S$ in $\mathbb{R}$ is the supremum of $S$ if and only if for every $\epsilon > 0$ there exists an $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$.

Proof.
Find the suprema and infima (if they exist) for the following sets.

1. $S_1 = \{1, 2, 3, 4, 5\}$
2. $S_2 = \{x : 0 \leq x \leq 1\}$
3. $S_3 = \{x : 0 < x < 1\}$
4. $S_4 = \{x : 0 < x\}$
Completeness Property of \( \mathbb{R} \): Every nonempty set of real numbers that has an upper bound also has a supremum in \( \mathbb{R} \). Likewise, every nonempty set of real numbers that has a lower bound also has an infimum in \( \mathbb{R} \).

Remark: The completeness of \( \mathbb{R} \) is essential to our later discussion of limits.

Remark: \( \mathbb{Q} \) is not complete.
Completeness Property of $\mathbb{R}$: Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$. Likewise, every nonempty set of real numbers that has an lower bound also has an infimum in $\mathbb{R}$.

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Remark: $\mathbb{Q}$ is not complete.
Read Section 2.3.

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Boxed problems should be written up separately and submitted for grading at class time on Friday.