Limits of Functions
MATH 464/506, *Real Analysis*

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Department of Mathematics

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Cluster Points

**Definition**

Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **cluster point** of $A$ if for every $\delta > 0$ there exists at least one point of $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

**Remarks:**

- This is equivalent to “A point $c \in \mathbb{R}$ is a cluster point of $A$ if every $\delta$-neighborhood $V_{\delta}(c)$ of $c$ contains at least one point of $A$ distinct from $c$.
- Point $c$ does not have to be a point in $A$.
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Theorem

A number $c \in \mathbb{R}$ is a cluster point of a subset $A$ of $\mathbb{R}$ if and only if there exists a sequence $(a_n)$ in $A$ such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

Proof.
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Let $A \subseteq \mathbb{R}$ and let $c$ be a cluster point of $A$. Suppose $f : A \rightarrow \mathbb{R}$ then a real number $L$ is said to be a limit of $f$ at $c$ if, given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Remarks:

- Notation: $\lim_{x \to c} f(x) = L$
- $\delta$ usually depends on $\epsilon$, therefore $\delta \equiv \delta(\epsilon)$
- $0 < |x - c| < \delta$ implies $x \neq c$
- If the limit of $f$ at $c$ does not exist we say $f$ diverges at $c$. 
Definition of the Limit

Definition

Let $A \subseteq \mathbb{R}$ and let $c$ be a cluster point of $A$. Suppose $f : A \to \mathbb{R}$ then a real number $L$ is said to be a **limit of $f$ at $c$** if, given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

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Theorem

If \( f : A \rightarrow \mathbb{R} \) and if \( c \) is a cluster point of \( A \), then \( f \) can have only one limit at \( c \).

Proof.

Theorem

Let \( f : A \rightarrow \mathbb{R} \) and let \( c \) be a cluster point of \( A \). Then the following statements are equivalent.

1. \( \lim_{x \to c} f(x) = L \)
2. Given any \( \epsilon \)-neighborhood \( V_\epsilon(L) \) of \( L \), there exists a \( \delta \)-neighborhood \( V_\delta(c) \) of \( c \) such that if \( x \neq c \) is any point in \( V_\delta(c) \cap A \), then \( f(x) \in V_\epsilon(L) \).
Uniqueness of Limits

Theorem

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If \( f : A \to \mathbb{R} \) and if \( c \) is a cluster point of \( A \), then \( f \) can have only one limit at \( c \).

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Proof.
Uniqueness of Limits

**Theorem**

*If* $f : A \rightarrow \mathbb{R}$ *and if* $c$ *is a cluster point of* $A$, *then* $f$ *can have only one limit at* $c$.

**Proof.**

Let $f : A \rightarrow \mathbb{R}$ *and let* $c$ *be a cluster point of* $A$. *Then the following statements are equivalent.*

1. $\lim_{x \to c} f(x) = L$

2. *Given any* $\epsilon$-*neighborhood* $V_\epsilon(L)$ *of* $L$, *there exists a* $\delta$-*neighborhood* $V_\delta(c)$ *of* $c$ *such that if* $x \neq c$ *is any point in* $V_\delta(c) \cap A$, *then* $f(x) \in V_\epsilon(L)$.

**Proof.**
How to prove \( \lim_{x \to c} f(x) = L \):

1. Let \( \epsilon > 0 \).
2. Find a value of \( \delta > 0 \) that will guarantee that whenever \( x \) is within a distance \( \delta \) from \( c \) (but not equal to \( c \)), \( f(x) \) is within a distance \( \epsilon \) of \( L \).
3. Prove that for this value of \( \delta \),

\[
\forall x \in D(f), \ 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.
\]
### Examples

<table>
<thead>
<tr>
<th>Example</th>
<th>Equation</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>( \lim_{x \to c} b = b )</td>
</tr>
<tr>
<td>2</td>
<td>( \lim_{x \to c} x = c )</td>
</tr>
<tr>
<td>3</td>
<td>( \lim_{x \to c} x^2 = c^2 )</td>
</tr>
<tr>
<td>4</td>
<td>( \lim_{x \to c} \frac{1}{x} = \frac{1}{c} ) if ( c &gt; 0 ).</td>
</tr>
<tr>
<td>5</td>
<td>( \lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5} )</td>
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Sequential Criterion for Limits

Theorem (Sequential Criterion)

Let $f : A \to \mathbb{R}$ and let $c$ be a cluster point of $A$. Then the following are equivalent.

1. $\lim_{x \to c} f(x) = L$.
2. For every sequence $(x_n)$ in $A$ that converges to $c$ such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to $L$.

Proof.
Theorem (Sequential Criterion)

Let $f : A \rightarrow \mathbb{R}$ and let $c$ be a cluster point of $A$. Then the following are equivalent.

1. $\lim_{x \rightarrow c} f(x) = L$.

2. For every sequence $(x_n)$ in $A$ that converges to $c$ such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to $L$.

Proof.
Divergence Criteria

Theorem

Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of $A$.

1. If $L \in \mathbb{R}$, then $f$ does not have limit $L$ at $c$ if and only if there exists a sequence $(x_n)$ in $A$ with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence $(x_n)$ converges to $c$ but the sequence $(f(x_n))$ does not converge to $L$.

2. The function $f$ does not have a limit at $c$ if and only if there exists a sequence $(x_n)$ in $A$ with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence $(x_n)$ converges to $c$ but the sequence $(f(x_n))$ does not converge in $\mathbb{R}$.
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<td>1</td>
<td>[ \lim_{x \to 0} \frac{1}{x} ]</td>
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<tr>
<td>2</td>
<td>[ \lim_{x \to 0} \begin{cases} +1 &amp; \text{if } x &gt; 0, \ 0 &amp; \text{if } x = 0, \ -1 &amp; \text{if } x &lt; 0. \end{cases} ]</td>
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<tr>
<td>3</td>
<td>[ \lim_{x \to 0} \sin \left( \frac{1}{x} \right) ]</td>
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Homework

- Read Section 4.1
- Page 104: 1, 3, 7, 9, 12, 14

Boxed problems should be written up separately and submitted for grading at class time on Friday.