A sequence \((a_n)\) is said to be

- **monotone increasing** if \(\forall n \in \mathbb{N}, a_n \leq a_{n+1}\); that is
  \[
a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots,
  \]

- **monotone decreasing** if \(\forall n \in \mathbb{N}, a_n \geq a_{n+1}\); that is
  \[
a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots,
  \]

- **strictly increasing** if \(\forall n \in \mathbb{N}, a_n < a_{n+1}\); that is
  \[
a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots,
  \]

- **strictly decreasing** if \(\forall n \in \mathbb{N}, a_n > a_{n+1}\); that is
  \[
a_1 > a_2 > \cdots > a_n > a_{n+1} > \cdots,
  \]
A sequence \((a_n)\) is said to be

- **monotone** if it is either **monotone increasing**, **monotone decreasing**, **strictly increasing**, or **strictly decreasing**,  
- **strictly monotone** if it is either **strictly increasing**, or **strictly decreasing**.
Convergence

Theorem (Monotone convergence theorem)

A monotone sequence of real numbers is convergent if and only if it is bounded. More precisely,

1. if \((a_n)\) is a monotone increasing sequence that is bounded above, then \(\lim_{n \to \infty} a_n = \sup\{a_n : n \in \mathbb{N}\}\),

2. if \((a_n)\) is a monotone decreasing sequence that is bounded below, then \(\lim_{n \to \infty} a_n = \inf\{a_n : n \in \mathbb{N}\}\),

Proof.
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Proof.
Harmonic Series

Example

Define the sequence $X = (x_n)$ to be

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$  

The sequence is increasing and unbounded and hence diverges.

$$x_{2n} = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \cdots + \left[\frac{1}{2^{n-1}} + 1 + \cdots + \frac{1}{2^n}\right]$$

$$> 1 + \frac{1}{2} + \left[\frac{1}{4} + \frac{1}{4}\right] + \cdots + \left[\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right]$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$

$$= 1 + \frac{n}{2}$$
Example

Let $a$ be any positive real number. Define the sequence $(x_n)$ inductively by

$$x_1 = \text{any positive real number}$$

$$\forall n \in \mathbb{N}, \quad x_{n+1} = \frac{x_n + \frac{a}{x_n}}{2}.$$

Then $(x_n)$ converges to a positive real number whose square is $a$. That is, $x_n \to \sqrt{a}$. Moreover, $\forall n \geq 2$,

$$0 \leq x_n - \sqrt{a} \leq \frac{x_n^2 - a}{\sqrt{a}}.$$
Euler's Number

Example

Define the sequence $X = (x_n)$ by $x_n = (1 + 1/n)^n$ for $n \in \mathbb{N}$. The sequence is bounded and increasing and hence converges (to $e$).
Homework

- Read Section 3.3.
- Pages 74–75: 1, 7, 10, 12, 13

Boxed problems should be written up separately and submitted for grading at class time on Friday.