Subsequences and Bolzano-Weierstrass Theorem

MATH 464/506, Real Analysis

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Suppose $X = (x_n)$ is a sequence. If $(n_k)$ is a strictly increasing sequence of natural numbers (i.e., $n_1 < n_2 < \cdots < n_k < \cdots$) then the sequence $X' = (x_{n_k})$ is said to be a subsequence of $X$ Thus $X'$ is the sequence

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, x_{n_{k+1}}, \ldots).$$
Theorem

If a sequence $X = (x_n)$ of real numbers converges to a real number $L$, then any subsequence $X' = (x_{n_k})$ of $X$ also converges to $L$.

Proof.
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Proof.
Equivalent Statements

Theorem

Let \( X = (x_n) \) be a sequence of real numbers. The following statements are equivalent:

1. The sequence \( X = (x_n) \) does not converge to \( L \in \mathbb{R} \).
2. There exists an \( \epsilon_0 > 0 \) such that for any \( k \in \mathbb{N} \), there exists \( n_k \in \mathbb{N} \) such that \( n_k \geq k \) and \( |x_{n_k} - L| \geq \epsilon_0 \).
3. There exists an \( \epsilon_0 > 0 \) and a subsequence \( X' = (x_{n_k}) \) of \( X \) such that \( |x_{n_k} - L| \geq \epsilon_0 \) for all \( k \in \mathbb{N} \).

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Proof.
Divergence Criteria

Theorem

If a sequence $X = (x_n)$ of real numbers has either of the following properties, then $X$ is divergent.

1. $X$ has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ whose limits are not equal.
2. $X$ is unbounded.
Theorem (Monotone Subsequence Theorem)

If $X = (x_n)$ is a sequence of real numbers, then there is a subsequence of $X$ that is monotone.

Proof.
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If \( X = (x_n) \) is a sequence of real numbers, then there is a subsequence of \( X \) that is monotone.

Proof.
Theorem (Bolzano-Weierstrass Theorem)

A bounded sequence of real numbers has a convergent subsequence.

Proof.

Let $X = (x_n)$ be a bounded sequence of real numbers and let $L \in \mathbb{R}$ have the property that every convergent subsequence of $X$ converges to $L$. Then the sequence $X$ converges to $L$. 
**Bolzano-Weierstrass Theorem**

**Theorem (Bolzano-Weierstrass Theorem)**

*A bounded sequence of real numbers has a convergent subsequence.*

**Proof.**

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Real Section 3.4.

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Boxed problems should be written up separately and submitted for grading at class time on Friday.