1 Euler Equations

A homogeneous **Euler Equation** is a second-order, linear ordinary differential equation of the form:

$$t^2u'' + \alpha t u' + \beta u = 0. \tag{1}$$

The expressions α and β are constants.

There are two popular ways to determine the solution to an Euler equation. The first is to assume that fundamental solutions have the form $u(t) = t^r$ where r is a constant. Differentiating this solution and substituting into eq. (1) produces

$$\begin{aligned} t^2 r(r-1)t^{r-2} + \alpha t r t^{r-1} + \beta t^r &= 0 \\ r(r-1)t^r + \alpha r t^r + \beta t^r &= 0 \\ r(r-1) + \alpha r + \beta &= 0 \quad (\text{assuming } t^r \neq 0) \\ r^2 + (\alpha - 1)r + \beta &= 0 \end{aligned} \tag{2}$$

Thus to solve an Euler equation, we would solve this quadratic equation for r.

As an alternative we may perform a change of variables by letting $t = e^z$. In this case

$$\begin{aligned} \frac{du}{dt} &= \frac{du}{dz}\frac{dz}{dt} = \frac{du}{dz}\frac{1}{t} \\ \frac{d^2u}{dt^2} &= \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d}{dt}\left(\frac{du}{dz}\frac{1}{t}\right) \\ &= \frac{d}{dt}\left(\frac{du}{dz}\right)\frac{1}{t} + \frac{du}{dz}\frac{d}{dt}\left(\frac{1}{t}\right) \\ &= \frac{d^2u}{dz^2}\frac{1}{t^2} - \frac{du}{dz}\frac{1}{t^2} \\ &= \left(\frac{d^2u}{dz^2} - \frac{du}{dz}\right)\frac{1}{t^2} \end{aligned}$$

Substituting in eq. (1) yields

$$\begin{aligned} \frac{d^2u}{dz^2} - \frac{du}{dz} + \alpha\frac{du}{dz} + \beta u &= 0 \\ \frac{d^2u}{dz^2} + (\alpha - 1)\frac{du}{dz} + \beta u &= 0 \end{aligned}$$

which is a second order, linear, constant coefficient ordinary differential equation. Its characteristic equation is the same as in eq. (2) above. To solve the Euler equation, we would once again solve the quadratic function for r and reverse the change of variables.

$$u = e^{rz} = (e^z)^r = t^r$$