Heat Equation in 3D

These notes will briefly outline the derivation of the heat equation in three dimensions. Throughout these notes the following quantities will be referenced.

c**:** specific heat of material, amount of heat per unit mass necessary to raise the temperature one degree

ρ**:** density of material, mass per unit volume

$u(x, y, z, t)$**:** temperature of the material at location (x, y, z) at time t

$Q(x, y, z, t)$**:** amount of heat energy generated per unit volume per unit time at location (x, y, z) at time t

$\boldsymbol{\phi}(x, y, z)$**:** heat energy flux at location (x, y, z)

K_0**:** thermal conductivity of the material, the amount of power per unit length per degree the material can conduct

k**:** thermal diffusivity of the material

We will also need Gauss's Theorem (Divergence Theorem):

Gauss's Theorem: Suppose region $R \subset \mathbb{R}^3$ is bounded by the closed surface S and that $\mathbf{n}(x, y, z)$ denotes the unit outward normal to S. If the components of the vector field $\mathbf{F}(x, y, z)$ have continuous first partial derivatives in R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_R \nabla \cdot \mathbf{F}\, dV.$$

Suppose the material in question occupies the region in three-dimensional space denoted by R and is bounded by the surface S. Let the unit normal vector to the surface S be denoted $\mathbf{n}$. The heat energy contained in R at time t is the value of

$$\iiint_R c\rho u\, dV.$$

The heat energy leaving region R through its boundary S is given by the surface integral

$$\iint_S \boldsymbol{\phi} \cdot \mathbf{n}\, dS.$$

The heat energy generated per unit time at time t in region R can be written as

$$\iiint_R Q\, dV.$$

Thus the conservation law of heat energy in region R can be expressed as

$$\begin{aligned}
\frac{d}{dt}\iiint\limits_R c\rho u\,dV &= -\iint\limits_S \boldsymbol{\phi}\cdot\mathbf{n}\,dS + \iiint\limits_R Q\,dV \\
&= -\iiint\limits_R \nabla\cdot\boldsymbol{\phi}\,dV + \iiint\limits_R Q\,dV \quad \text{(Gauss's Theorem)} \\
\iiint\limits_R c\rho\frac{\partial u}{\partial t}\,dV &= \iiint\limits_R \left(Q - \nabla\cdot\boldsymbol{\phi}\right)\,dV \\
0 &= \iiint\limits_R \left(Q - \nabla\cdot\boldsymbol{\phi} - c\rho\frac{\partial u}{\partial t}\right)\,dV
\end{aligned}$$

Since R is an arbitrary three-dimensional region then

$$c\rho\frac{\partial u}{\partial t} + \nabla\cdot\boldsymbol{\phi} - Q = 0. \tag{1}$$

In three dimensions Fourier's Law of Heat Conduction can be stated as

$$\boldsymbol{\phi} = -K_0\nabla u.$$

Substituting this into eq. (1) yields

$$\begin{aligned}
c\rho\frac{\partial u}{\partial t} + \nabla\cdot(-K_0\nabla u) - Q &= 0 \\
c\rho\frac{\partial u}{\partial t} &= K_0\left(\nabla\cdot\nabla u\right) + Q \\
&= K_0\nabla^2 u + Q \\
\frac{\partial u}{\partial t} &= k\nabla^2 u + \frac{Q}{c\rho}
\end{aligned}$$

If there are no sources of heat in region R then the heat equation simplifies to

$$\frac{\partial u}{\partial t} = k\nabla^2 u.$$