Problem 2.4.1

Solve

\[ u_t = ku_{xx} \quad \text{for } 0 < x < L, \ t > 0, \]
\[ u_x(0, t) = 0 \quad \text{for } t > 0, \]
\[ u_x(L, t) = 0 \quad \text{for } t > 0, \]

subject to the following initial conditions:

(a) \[ u(x, 0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases} \]

Since the boundary conditions are of Neumann type (zero flux at the ends of the rod), the solution to the heat equation is (according to equation 2.4.19 on page 61)

\[ u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-kn^2 \pi^2 t / L^2} \cos \frac{n \pi x}{L} \]

where

\[ A_0 = \frac{1}{L} \int_0^L u(x, 0) \, dt \]
\[ A_n = \frac{2}{L} \int_0^L u(x, 0) \cos \frac{n \pi x}{L} \, dx \quad (\text{if } n \in \mathbb{N}) \]

Thus

\[ A_0 = \frac{1}{L} \int_0^L u(x, 0) \, dx = \frac{1}{L} \int_{L/2}^L 1 \, dx = \frac{1}{L} L \left| x \right|_{L/2}^{L/2} = \frac{1}{2} \]
and
\[ A_n = \frac{2}{L} \int_0^L u(x, 0) \cos \frac{n \pi x}{L} \, dx = \frac{2}{L} \int_{L/2}^L \cos \frac{n \pi x}{L} \, dx = \frac{2}{L} \cdot \frac{L}{n \pi} \sin \frac{n \pi}{L} \bigg|_{L/2} = \frac{2}{n \pi} \left( \sin \frac{n \pi}{2} - \sin \frac{n \pi}{2} \right) = -\frac{2}{n \pi} \sin \frac{n \pi}{2} \quad (\text{if } n \in \mathbb{N}) \]

Thus the solution can be expressed as
\[ u(x, t) = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{n \pi}{2} \right) e^{-k n^2 \pi^2 t / L^2} \cos \frac{n \pi x}{L}. \]

(b) \( u(x, 0) = 6 + 4 \cos \frac{3 \pi x}{L} \)

Since the boundary conditions are of Neumann type (zero flux at the ends of the rod), the solution to the heat equation is (according to equation 2.4.19 on page 61)
\[ u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k n^2 \pi^2 t / L^2} \cos \frac{n \pi x}{L} \]

where
\[ A_0 = \frac{1}{L} \int_0^L u(x, 0) \, dt \quad A_n = \frac{2}{L} \int_0^L u(x, 0) \cos \frac{n \pi x}{L} \, dx \quad (\text{if } n \in \mathbb{N}) \]

In this case the initial condition is a finite linear combination of eigenfunctions of the boundary value problem and thus we may simply equate coefficients rather than using the Fourier-Euler formulas for the coefficients. Here we need
\[ A_0 = 6 \quad A_3 = 4 \quad A_n = 0 \quad (\text{for } n \neq 0 \text{ and } n \neq 3). \]

Thus the solution to the heat equation may be expressed as
\[ u(x, t) = 6 + 3e^{-9 \pi^2 k t / L^2} \cos \frac{3 \pi x}{L}. \]
(c) \( u(x, 0) = -2 \sin \frac{\pi x}{L} \)

Since the boundary conditions are of Neumann type (zero flux at the ends of the rod), the solution to the heat equation is (according to equation 2.4.19 on page 61)

\[
    u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\kappa n^2\pi^2 t / L^2} \cos \frac{n\pi x}{L}
\]

where

\[
    A_0 = \frac{1}{L} \int_0^L u(x, 0) \, dx
\]

\[
    A_n = \frac{2}{L} \int_0^L u(x, 0) \cos \frac{n\pi x}{L} \, dx \quad \text{(if } n \in \mathbb{N})
\]

Thus

\[
    A_0 = \frac{1}{L} \int_0^L u(x, 0) \, dx = \frac{1}{L} \int_0^L (-2 \sin \frac{\pi x}{L}) \, dx = -\frac{2}{L} \left( -\frac{L}{\pi} \right) \cos \frac{\pi x}{L} \bigg|_0^L = \frac{2}{\pi} (\cos \pi - \cos 0) = -\frac{4}{\pi}
\]

and

\[
    A_n = \frac{2}{L} \int_0^L \left( -2 \sin \frac{\pi x}{L} \right) \cos \frac{n\pi x}{L} \, dx
\]

\[
    = -\frac{4}{L} \int_0^L \sin \frac{\pi x}{L} \cos \frac{n\pi x}{L} \, dx
\]

\[
    = -\frac{2}{L} \int_0^L \sin \left( \frac{\pi x}{L} + \frac{n\pi x}{L} \right) + \sin \left( \frac{\pi x}{L} - \frac{n\pi x}{L} \right) \, dx
\]

\[
    = -\frac{2}{L} \int_0^L \sin \frac{\pi}{L}(1+n)x + \sin \frac{\pi}{L}(1-n)x \, dx
\]

When \( n = 1 \) then

\[
    A_1 = -\frac{2}{L} \int_0^L \sin \frac{2\pi x}{L} \, dx = 0.
\]

When \( n > 1 \) we have

\[
    A_n = \frac{2}{L} \left[ \frac{L}{(1+n)\pi} \cos \frac{\pi}{L}(1+n)x + \frac{L}{(1-n)\pi} \cos \frac{\pi}{L}(1-n)x \right]_0^L
\]

\[
    = \frac{2}{\pi} \left[ \frac{1}{1+n} \cos \frac{\pi}{L}(1+n)x + \frac{1}{1-n} \cos \frac{\pi}{L}(1-n)x \right]_0^L
\]

\[
    = \frac{2}{\pi} \left[ \frac{1}{1+n} (\cos(1+n)\pi - 1) + \frac{1}{1-n} (\cos(1-n)\pi - 1) \right]
\]

\[
    = \frac{2}{\pi} \left[ \frac{1}{1+n} ((-1)^{1+n} - 1) + \frac{1}{1-n} ((-1)^{1-n} - 1) \right]
\]
We can see that if \( n \) is an odd natural number then \( A_n = 0 \). When the subscript is an even natural number
\[
A_{2n} = \frac{2}{\pi} \left[ \frac{-2}{1+2n} + \frac{-2}{1-2n} \right] = \frac{8}{\pi(4n^2 - 1)}.
\]
Thus the solution to the heat equation may be expressed as
\[
u(x, t) = -\frac{4}{\pi} + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} e^{-4n^2\pi^2kt/L^2} \cos \frac{2n\pi x}{L}.
\]

(d) \( u(x, 0) = -3 \cos \frac{8\pi x}{L} \)

Since the boundary conditions are of Neumann type (zero flux at the ends of the rod), the solution to the heat equation is (according to equation 2.4.19 on page 61)
\[
u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2t/L^2} \cos \frac{n\pi x}{L}
\]
where
\[
A_0 = \frac{1}{L} \int_0^L u(x, 0) \, dt
A_n = \frac{2}{L} \int_0^L u(x, 0) \cos \frac{n\pi x}{L} \, dx \quad \text{(if } n \in \mathbb{N})
\]
In this case the initial condition is a finite linear combination of eigenfunctions of the boundary value problem and thus we may simply equate coefficients rather than using the Fourier-Euler formulas for the coefficients. Here we need
\[
A_8 = -3
A_n = 0 \quad \text{for } n \neq 8.
\]
Thus the solution to the heat equation may be expressed as
\[
u(x, t) = -3e^{-64\pi^2kt/L^2} \cos \frac{8\pi x}{L}.
\]

**Problem 2.4.2**

Solve
\[
u_t = ku_{xx} \quad \text{for } 0 < x < L, \ t > 0,

\nu(x, 0) = f(x) \quad \text{for } 0 < x < L,

\nu_x(0, t) = 0 \quad \text{for } t > 0,

\nu(L, t) = 0 \quad \text{for } t > 0.
\]
For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

Using separation of variables we assume that

\[ u(x, t) = X(x)T(t). \]

Differentiating and substituting into the PDE produces

\[ \frac{1}{k} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2. \]

The eigenvalue/eigenfunction problem is then

\[ X'' + \lambda X = 0 \quad \text{with} \quad X'(0) = 0 \quad \text{and} \quad X(L) = 0. \]

If \( \lambda = 0 \) we have \( X(x) = c_1 x + c_2 \). The boundary condition at \( x = 0 \) implies \( X'(0) = 0 = c_1 \). The boundary condition at \( x = L \) implies \( X(L) = c_2 = 0 \). Thus when \( \lambda = 0 \) there are no non-trivial solutions.

If \( \lambda^2 < 0 \) then solution will grow in time contradicting one of the assumptions of the exercise. Therefore we need only consider \( \lambda > 0 \). In this case

\[
\begin{align*}
X(x) &= c_1 \cos \lambda x + c_2 \sin \lambda x, \\
X'(x) &= -c_1 \lambda \sin \lambda x + c_2 \lambda \cos \lambda x.
\end{align*}
\]

The boundary condition at \( x = 0 \) implies \( X'(0) = 0 = c_2 \lambda \) which in turn implies that \( c_2 = 0 \). The boundary condition at \( x = L \) implies \( X(L) = c_1 \cos \lambda L = 0 \). To avoid the situation of having only the trivial solution we must assume \( c_1 \neq 0 \) and thus that

\[
\cos \lambda L = 0 \quad \text{which implies} \quad \lambda_n = \frac{(2n - 1)\pi}{2L},
\]

where \( n = 1, 2, \ldots \). Thus the eigenvalues of the boundary value problem are

\[
\left\{ \frac{(4n^2 - 4n + 1)\pi^2}{4L^2} \right\} \quad \text{for} \quad n = 1, 2, \ldots,
\]

with corresponding eigenfunctions

\[
X_n(x) = \cos \left( \frac{(2n - 1)\pi x}{2L} \right).
\]

Thus the solution to the boundary value problem will be of the form

\[
\begin{align*}
u(x, t) &= \sum_{n=1}^{\infty} A_n e^{-\frac{(4n^2 - 4n + 1)x^2}{4L^2}} \cos \left( \frac{(2n - 1)\pi x}{2L} \right).
\end{align*}
\]
To satisfy the initial condition we must have

\[ u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \cos \left( \frac{(2n-1)\pi x}{2L} \right). \]

Multiply the last equation by the \( m \)th eigenfunction and integrate the product from \( x = 0 \) to \( x = L \).

\[
\int_0^L f(x) \cos \left( \frac{(2m-1)\pi x}{2L} \right) \, dx = \sum_{n=1}^{\infty} A_n \int_0^L \cos \left( \frac{(2n-1)\pi x}{2L} \right) \cos \left( \frac{(2m-1)\pi x}{2L} \right) \, dx = A_m \frac{L}{2}
\]

Hence the coefficients of the infinite series are given by the formula:

\[ A_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{(2n-1)\pi x}{2L} \right) \, dx \]

for \( n = 1, 2, \ldots \).

**Exercise 2.4.4**

There are no negative eigenvalues for the boundary value problem

\[
\frac{d^2 \phi}{dx^2} = -\lambda \phi \quad \text{subject to} \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0.
\]

To prove this suppose \( \lambda = -\alpha^2 \) with \( \alpha \in \mathbb{R} \) and \( \alpha \neq 0 \) and suppose

\[
\frac{d^2 \phi}{dx^2} = -\lambda \phi = \alpha^2 \phi
\]

then

\[
\frac{d^2 \phi}{dx^2} - \alpha^2 \phi = 0
\]

which implies

\[
\phi(x) = Ae^{\alpha x} + Be^{-\alpha x} \quad \text{and} \quad \phi'(x) = A\alpha e^{\alpha x} - B\alpha e^{-\alpha x}
\]

The boundary condition \( \phi'(0) = (A - B)\alpha = 0 \) implies that \( A = B \). The boundary condition \( \phi'(L) = 0 \) means

\[
0 = A\alpha e^{\alpha L} - B\alpha e^{-\alpha L} = A\alpha (e^{\alpha L} - e^{-\alpha L}) = 2A\alpha \sinh(\alpha L)
\]

which implies that \( A = B = 0 \). Thus there are no nontrivial solutions when \( \lambda \) is negative.
Exercise 2.4.6

Find the equilibrium temperature distribution for the thin circular ring described by the boundary value problem:

\[
\begin{align*}
    u_t &= ku_{xx} \quad \text{for } -L < x < L, \ t > 0, \\
    u(x, 0) &= f(x) \quad \text{for } -L < x < L, \\
    u(-L, t) &= u(L, t) \quad \text{for } t > 0, \\
    u_x(-L, t) &= u_x(L, t) \quad \text{for } t > 0
    \end{align*}
\]

(a) Directly from the equilibrium problem

At equilibrium \( \frac{\partial u}{\partial t} = 0 \) which implies \( u(x) = Ax + B \). The only linear function satisfying the boundary conditions is a constant

\[
u(x) = B = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx.
\]

(b) Computing the limit \( t \to \infty \)

According to equation (2.4.38)

\[
u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 kt/L^2} \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 kt/L^2} \sin \frac{n\pi x}{L}.
\]

Thus

\[
\lim_{t \to \infty} u(x, t) = a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx
\]

according to equation (2.4.43).