Please work the following problems for homework and return them to me by 4:00PM on Monday, April 10, 2006. Each problem is worth 10 points unless marked otherwise.

1. Determine the eigenvalues and eigenfunctions for the boundary value problem,

\[
\phi''(x) + \lambda^2 \phi(x) = 0 \\
\phi'(0) = 0 \\
\phi'(1) + \phi(1) = 0
\]

2. Express the function \( f(x) = 1 - x \) as a series of eigenfunctions for the eigenfunctions \( \phi_n(x) \) found in the previous exercise.

3. Determine the eigenvalues and eigenfunctions for the boundary value problem,

\[
\phi''(x) - 2\phi'(x) + (1 + \lambda)\phi(x) = 0 \\
\phi(0) = 0 \\
\phi(1) = 0
\]

You may find it easier to solve this ODE if you perform the change of dependent variable \( \phi(x) = s(x)u(x) \). Determine a form for \( s(x) \) so that the ODE contains no \( u'(x) \) term.

4. Consider the Sturm-Liouville problem

\[
\frac{d}{dx} \left( p(x) \frac{d\phi}{dx} \right) + q(x)\phi + \lambda \sigma(x) \phi = 0 \\
a_1 \phi(0) + a_2 \phi'(0) = 0 \\
b_1 \phi(1) + b_2 \phi'(1) = 0
\]

(a) Show that if \( \lambda \) is an eigenvalue and \( \phi(x) \) is its corresponding eigenfunction, then

\[
\lambda \int_0^1 \sigma(x)\phi^2(x) \, dx = \int_0^1 \left( p(x)(\phi'(x))^2 - q(x)\phi^2(x) \right) \, dx + \frac{b_1}{b_2} p(1)\phi^2(1) - \frac{a_1}{a_2} p(0)\phi^2(0)
\]

provided that \( a_2 \neq 0 \) and \( b_2 \neq 0 \).

(b) Show that if \( q(x) \leq 0 \) and if \( b_1/b_2 \geq 0 \) and \( a_1/a_2 \leq 0 \) then \( \lambda \geq 0 \).

(c) Show that if \( q(x) \leq 0 \) on \( [0,1] \) with \( q(x) < 0 \) for some \( x \in [0,1] \) and \( a_1 = b_1 = 0 \) then \( \lambda > 0 \).