Please work the following problems for homework and turn them in at class time on Thursday, January 31, 2008. Each problem is worth 10 points unless marked otherwise.

1. Consider the linear homogeneous heat equation for a one-dimensional bar of length $L = 1$:

$$u_t = k^2 u_{xx}.$$ 

Assume that the bar is insulated along its length and that the ends of the bar are kept at constant temperature $u(0, t) = u(1, t) = 0$. Assume that the initial temperature distribution along the bar is given by the function

$$u(x, 0) = f(x) = \begin{cases} 
0 & \text{if } 0 < x < \frac{2}{5}, \\
100 & \text{if } \frac{2}{5} < x < \frac{3}{5}, \\
0 & \text{if } \frac{3}{5} < x < 1.
\end{cases}$$

(a) Find the eigenfunctions and eigenvalues for the boundary value problem.

Since the boundary conditions are of Dirichlet type with $L = 1$ we have

Eigenfunctions: $X_n(x) = \sin(n\pi x)$

Eigenvalues: $\lambda_n^2 = n^2\pi^2$

for $n \in \mathbb{N}$.

(b) Find the Fourier series solution to the initial value problem.

Using the Fourier-Euler integral formula:

$$A_n = 2 \int_0^1 f(x) \sin(n\pi x) \, dx$$

$$= 2 \int_{\frac{2}{5}}^{\frac{3}{5}} 100 \sin(n\pi x) \, dx$$

$$= \frac{200}{n\pi} \left( \cos\left(\frac{2n\pi}{5}\right) - \cos\left(\frac{3n\pi}{5}\right) \right)$$

for $n \in \mathbb{N}$. Thus

$$u(x, t) = 200 \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n\pi}{5}\right) - \cos\left(\frac{3n\pi}{5}\right)}{n\pi} e^{-n^2k^2\pi^2t} \sin(n\pi x).$$

(c) Assuming $k^2 = 0.20$ plot the graph of $u(x, 0.10)$ for $0 < x < 1$ using the first three non-zero terms of the Fourier series.
2. Re-work the first exercise if the ends of the metal rod are insulated so that the boundary condition is \( u_x(0, t) = u_x(1, t) = 0 \).

(a) Find the eigenfunctions and eigenvalues for the boundary value problem.

Since the boundary conditions are of Neumann type with \( L = 1 \) we have

\[
\begin{align*}
\text{Eigenfunctions: } & X_n(x) = \cos(n\pi x) \\
\text{Eigenvalues: } & \lambda_n^2 = \frac{n^2\pi^2}{L^2}
\end{align*}
\]

for \( n = 0, 1, \ldots \).

(b) Find the Fourier series solution to the initial value problem.

Using the Fourier-Euler integral formula:

\[
A_n = 2 \int_0^1 f(x) \cos(n\pi x) \, dx
\]

\[
= 2 \int_{3/5}^{2/5} 100 \cos(n\pi x) \, dx
\]

\[
= \frac{200}{n\pi} \left( \sin \left( \frac{3n\pi}{5} \right) - \sin \left( \frac{2n\pi}{5} \right) \right)
\]

for \( n \in \mathbb{N} \).

\[
A_0 = \int_0^1 f(x) \, dx
\]

\[
= \int_{3/5}^{2/5} 100 \, dx
\]

\[
= 20
\]

Thus

\[
u(x, t) = 20 + 200 \sum_{n=1}^{\infty} \frac{\sin \left( \frac{3n\pi}{5} \right) - \sin \left( \frac{2n\pi}{5} \right)}{n\pi} e^{-n^2\pi^2 t} \cos(n\pi x).
\]
(c) Assuming \( k^2 = 0.20 \) plot the graph of \( u(x, 0.10) \) for \( 0 < x < 1 \) using the first three non-zero terms of the Fourier series.

\[
\begin{array}{c}
\text{Graph showing the function } u(x) \text{ for } 0 < x < 1.
\end{array}
\]

(d) What is the steady-state solution to this problem?

\[
\lim_{t \to \infty} u(x, t) = 20.
\]