1. Consider the function:

\[ f(x) = \begin{cases} 
2(x + 1) & \text{if } -1 \leq x \leq 0, \\
x & \text{if } 0 < x \leq 1.
\end{cases} \]

(a) Sketch the graph of the Fourier Series representation of \( f(x) \) on the interval \([-4, 4]\).

(b) Sketch the graph of the Fourier Sine Series representation of \( f(x) \) on the interval \([-4, 4]\).

(c) Sketch the graph of the Fourier Cosine Series representation of \( f(x) \) on the interval \([-4, 4]\).

2. Using trigonometric identities (e.g. product to sum formulas and half-angle identities) find the Fourier Series of the following functions without computing any integrals.

(a) \( f(x) = \cos^2(\pi x) \sin^2(\pi x) \) for \(-1 \leq x \leq 1\).
\[ f(x) = \cos^2(\pi x) \sin^2(\pi x) \]
\[ = \frac{1}{4} (1 + \cos(2\pi x))(1 - \cos(2\pi x)) \]
\[ = \frac{1}{4} (1 - \cos^2(2\pi x)) \]
\[ = \frac{1}{4} (1 - \frac{1}{2}(1 + \cos(4\pi x))) \]
\[ = \frac{1}{8} - \frac{1}{8} \cos(4\pi x) \]

(b) \( g(x) = \sin(x) [\sin(x) + \cos(x)]^2 \) for \(-\pi \leq x \leq \pi \).

\[ g(x) = \sin(x) [\sin(x) + \cos(x)]^2 \]
\[ = \sin(x) [\sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x)] \]
\[ = \sin(x) [1 + \sin(2x)] \]
\[ = \sin(x) + \sin(x)\sin(2x) \]
\[ = \sin(x) + \frac{1}{2} \cos(x) - \frac{1}{2} \cos(3x) \]

3. Consider the function

\[ f(x) = \begin{cases} 
1 + x & \text{if} -1 \leq x \leq 0, \\
0 & \text{if} 1 < |x| \leq 2, \\
1 - x & \text{if} 0 \leq x \leq 1.
\end{cases} \]

Find the Fourier Series for this function.

The function is an even function, thus we need only find the Fourier Cosine Series for it.

\[ A_0 = \frac{1}{4} \int_{-2}^{2} f(x) \, dx \]
\[ = \frac{1}{4} \left( \int_{-1}^{0} (1 + x) \, dx + \int_{0}^{1} (1 - x) \, dx \right) \]
\[ = \frac{1}{4} \left( x + \frac{x^2}{2} \right|_{-1}^{0} + \frac{1}{4} \left( x - \frac{x^2}{2} \right|_{0}^{1} \right) \]
\[ = \frac{1}{8} + \frac{1}{8} \]
\[ = \frac{1}{4} \]
For \( n \in \mathbb{N} \),

\[
A_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n\pi x}{2} \, dx
\]

\[
= \frac{1}{2} \int_{-1}^{0} (1 + x) \cos \frac{n\pi x}{2} \, dx + \frac{1}{2} \int_{0}^{1} (1 - x) \cos \frac{n\pi x}{2} \, dx
\]

\[
= \frac{4}{n^2\pi^2} \sin^2 \frac{n\pi}{4} + \frac{4}{n^2\pi^2} \sin^2 \frac{3n\pi}{4}
\]

\[
= \frac{8}{n^2\pi^2} \sin^2 \frac{n\pi}{4}
\]

\[
= \begin{cases} 
0 & \text{if } n = 4m, \\
1/2 & \text{if } n = 4m + 1, \\
1 & \text{if } n = 4m + 2, \\
1/2 & \text{if } n = 4m + 3.
\end{cases}
\]

Thus

\[
f(x) \sim \frac{1}{4} + \frac{4}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(4m+1)^2} \cos \frac{(4m+1)\pi x}{2}
\]

\[
+ \frac{2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m+1)^2} \cos(2m+1)\pi x
\]

\[
+ \frac{4}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(4m+3)^2} \cos \frac{(4m+3)\pi x}{2}.
\]

4. Find the Fourier Series for the derivative of the function given in Ex. 3.

Since \( f(x) \) is piecewise smooth and \( f(-2) = f(2) \) then we may differentiate its Fourier Series term-by-term.

\[
f'(x) \sim -\frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{4m+1} \sin \frac{(4m+1)\pi x}{2} - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m+1} \sin(2m+1)\pi x
\]

\[
- \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{4m+3} \sin \frac{(4m+3)\pi x}{2}.
\]