Please work the following problems for homework and turn them in at class time on Thursday, March 27, 2008. Each problem is worth 10 points unless marked otherwise.

1. Put the following ordinary differential equations and boundary conditions into Sturm-Liouville form. You do not have to solve the boundary value problems.

(a) \( xy'' + y' + \lambda y = 0, \ y(0) = 0, \ y(1) = 0 \)
(b) \( xy'' + 2y' + \lambda y = 0, \ y(1) = 0, \ y'(2) = 0 \)
(c) \( xy'' - y' + \lambda xy = 0, \ y(0) = 0, \ y(1) = 0 \)
(d) \( y'' + \lambda xy = 0, \ y(-1) = 0, \ y(1) = 0 \)
(e) \( (1-x^2)y'' - 2xy' + \lambda y = 0, \ y(-1) = 0, \ y(1) = 0 \)

2. Determine the eigenvalues and eigenfunctions of the following Sturm-Liouville boundary value problem.

\[
\begin{align*}
y'' + \lambda y &= 0 \\
y(0) + y'(0) &= 0 \\
y(1) + y'(1) &= 0
\end{align*}
\]

3. The second order, linear ordinary differential equation

\[
(1-x^2)y'' - xy' + n^2y = 0
\]

defined for \(-1 < x < 1\), where \(n = 0, 1, 2, \ldots\) is known as Chebyshev’s equation. Consider the boundary conditions: \(y(1) = 1\) and \(y'(1)\) is finite.

(a) Put this equation in Sturm-Liouville form.

(b) Use power series techniques to show that for each \(n\), Chebyshev’s equation has one polynomial solution of degree \(n\). These are called Chebyshev polynomials and are usually denoted \(T_n(x)\).

(c) Show that the Chebyshev polynomials are orthogonal on \((-1, 1)\) with respect to the weight function

\[
\sigma(x) = \frac{1}{\sqrt{1-x^2}}.
\]