Please work the following problems for homework and turn them in at class time on Thursday, March 27, 2008. Each problem is worth 10 points unless marked otherwise.

1. Put the following ordinary differential equations and boundary conditions into Sturm-Liouville form. You do not have to solve the boundary value problems.

(a) \( xy'' + y' + \lambda y = 0 \), \( y(0) = 0 \), \( y(1) = 0 \)

\[
\begin{align*}
(xy')' + \lambda y &= 0 \\
y(0) &= 0 \\
y(1) &= 0
\end{align*}
\]

(b) \( xy'' + 2y' + \lambda y = 0 \), \( y(1) = 0 \), \( y'(2) = 0 \)

If we multiply the ODE by \( x \) we get

\[
x^2y'' + 2xy' + \lambda xy = 0
\]

which can be written in the Sturm-Liouville form:

\[
\begin{align*}
(x^2y')' + \lambda xy &= 0 \\
y(0) &= 0 \\
y'(2) &= 0
\end{align*}
\]

(c) \( xy'' - y' + \lambda xy = 0 \), \( y(0) = 0 \), \( y(1) = 0 \)

If we multiply the ODE by \( 1/x^2 \) we get

\[
\frac{1}{x}y'' - \frac{1}{x^2}y' + \lambda \left(\frac{1}{x}\right)y = 0
\]

which can be written in the Sturm-Liouville form:

\[
\begin{align*}
\left(\frac{1}{x}y'\right)' + \lambda \left(\frac{1}{x}\right)y &= 0 \\
y(0) &= 0 \\
y(1) &= 0
\end{align*}
\]
(d) \( y'' + \lambda xy = 0, \ y(-1) = 0, \ y(1) = 0 \)

\[
\begin{align*}
(y')' + \lambda xy &= 0 \\
y(-1) &= 0 \\
y(1) &= 0
\end{align*}
\]

Note that \( \sigma(x) = x \) is not positive on all of \([-1, 1]\).

(e) \((1 - x^2)y'' - 2xy' + \lambda y = 0, \ y(-1) = 0, \ y(1) = 0 \)

\[
\begin{align*}
((1 - x^2)y')' + \lambda y &= 0 \\
y(-1) &= 0 \\
y(1) &= 0
\end{align*}
\]

2. Determine the eigenvalues and eigenfunctions of the following Sturm-Liouville boundary value problem.

\[
\begin{align*}
y'' + \lambda y &= 0 \\
y(0) + y'(0) &= 0 \\
y(1) + y'(1) &= 0
\end{align*}
\]

When \( \lambda > 0 \) then the solution to the ODE is

\[y(x) = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x.\]

The first boundary condition implies

\[B + A\sqrt{\lambda} = 0\]

which means \( B = -\sqrt{\lambda}A \). Substituting this into the solution and using the second boundary condition we have

\[
A \sin \sqrt{\lambda} - A\sqrt{\lambda} \cos \sqrt{\lambda} + A\sqrt{\lambda} \cos \sqrt{\lambda} + A\lambda \sin \sqrt{\lambda} = 0
\]

\[A(1 + \lambda) \sin \sqrt{\lambda} = 0\]

Thus

\[\lambda_n = n^2 \pi^2\]

for \( n \in \mathbb{N} \).

If \( \lambda = 0 \), then \( y = Ax + B \) and the boundary conditions imply

\[
\begin{align*}
B + A &= 0 \\
2A + B &= 0
\end{align*}
\]
which means $A = B = 0$. Thus $\lambda = 0$ is not an eigenvalue.

Suppose now that $\lambda < 0$ then the solution to the ODE is

$$y(x) = A \sinh \sqrt{-\lambda} x + B \cosh \sqrt{-\lambda} x.$$ 

The first boundary condition implies

$$B + A \sqrt{-\lambda} = 0$$

which means $B = -\sqrt{-\lambda} A$. Substituting this into the solution and using the second boundary condition we have

$$A \sinh \sqrt{-\lambda} + A \sqrt{-\lambda} \cosh \sqrt{-\lambda} - A \sqrt{-\lambda} \cosh \sqrt{-\lambda} + A \lambda \sinh \sqrt{-\lambda} = 0$$

$$A (1 + \lambda) \sinh \sqrt{-\lambda} = 0$$

which implies $\lambda_{-1} = -1$.

We may summarize the eigenfunctions and eigenvalues in the following table.

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Eigenfunction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{-1} = -1$</td>
<td>$\sinh x - \cosh x$</td>
</tr>
<tr>
<td>$\lambda_n = n^2 \pi^2$</td>
<td>$\sin(n \pi x) - n \pi \cos(n \pi x)$</td>
</tr>
</tbody>
</table>

3. The second order, linear ordinary differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

defined for $-1 < x < 1$, where $n = 0, 1, 2, \ldots$ is known as Chebyshev’s equation. Consider the boundary conditions: $y(1) = 1$ and $y'(1)$ is finite.

(a) Put this equation in Sturm-Liouville form.

If we divide the ordinary differential equation by $\sqrt{1 - x^2}$ we obtain

$$\sqrt{1 - x^2}y'' - \frac{x}{\sqrt{1 - x^2}}y' + \frac{n^2}{\sqrt{1 - x^2}}y = 0.$$ 

In Sturm-Liouville form this becomes

$$\left( \sqrt{1 - x^2} y' \right)' + \frac{n^2}{\sqrt{1 - x^2}} y = 0$$

$$y(1) = 1$$

$$|y'(1)| < \infty$$
(b) Use power series techniques to show that for each \( n \), Chebyshev’s equation has one polynomial solution of degree \( n \). These are called Chebyshev polynomials and are usually denoted \( T_n(x) \).

Assume the series solution to Chebyshev’s ODE has the form \( y(x) = \sum_{k=0}^{\infty} a_k x^k \). Differentiating this solution and substituting into Chebyshev’s equation yields the following.

\[
0 = (1 - x^2) \sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - x \sum_{k=0}^{\infty} ka_k x^{k-1} + n^2 \sum_{k=0}^{\infty} a_k x^k
\]

\[
= \sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1)a_k x^{k-1} - \sum_{k=0}^{\infty} ka_k x^k + n^2 \sum_{k=0}^{\infty} a_k x^k
\]

\[
= \sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} [k(k-1) + k - n^2]a_k x^k
\]

\[
= \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} [k^2 - n^2]a_k x^k
\]

\[
= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k - \sum_{k=0}^{\infty} [k^2 - n^2]a_k x^k
\]

\[
= \sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} - (k^2 - n^2) a_k] x^k
\]

Thus we may treat \( a_0 \) and \( a_1 \) as arbitrary and for \( k \geq 0 \) the recursion relation among the coefficients is

\[
a_{k+2} = \frac{k^2 - n^2}{(k+1)(k+2)} a_k.
\]

If \( n \) is even and if we let \( a_0 = 1 \) and \( a_1 = 0 \), then

\[
y_1(x) = 1 - \frac{n^2}{2} x^2 - \frac{n^2(2^2 - n^2)}{4!} x^4 - \frac{n^2(2^2 - n^2)(4^2 - n^2)}{6!} x^6 - \ldots - \frac{n^2(2^2 - n^2) \cdots ((n-2)^2 - n^2)}{n!} x^n.
\]

Note that \( y_1(x) \) is a polynomial of degree \( n \).

If \( n \) is odd and if we let \( a_0 = 0 \) and \( a_1 = 1 \), then

\[
y_2(x) = x + \frac{1-n^2}{3!} x^3 + \frac{(1-n^2)(3^2 - n^2)}{5!} x^5 + \frac{(1-n^2)(3^2 - n^2)(5^2 - n^2)}{7!} x^7 + \ldots + \frac{(1-n^2)(3^2 - n^2) \cdots ((n-2)^2 - n^2)}{n!} x^n.
\]

Note that \( y_2(x) \) is a polynomial of degree \( n \).
(c) Show that the Chebyshev polynomials are orthogonal on $(-1, 1)$ with respect to the weight function

$$\sigma(x) = \frac{1}{\sqrt{1 - x^2}}.$$ 

Let the linear operator $L[\cdot]$ be defined as

$$L[u] = \left(\sqrt{1 - x^2}u'\right)'$$

then the Sturm-Liouville form of Chebyshev's differential equation can be written as

$$L[y] + \frac{n^2}{\sqrt{1 - x^2}}y = 0.$$ 

Suppose that $y_n(x)$ and $y_m(x)$ are two eigenfunctions corresponding to the eigenvalues $n^2$ and $m^2$ respectively with $n^2 \neq m^2$. The Green’s formula states that

$$\int_{-1}^{1} (y_n(x)L[y_m] - y_m(x)L[y_n]) \, dx = \left. \sqrt{1 - x^2} (y_n(x)y_m'(x) - y_m(x)y_n'(x)) \right|_{-1}^{1}$$

$$\int_{-1}^{1} \left( -\frac{m^2 y_n(x)y_m(x)}{\sqrt{1 - x^2}} + \frac{n^2 y_n(x)y_m(x)}{\sqrt{1 - x^2}} \right) \, dx = 0$$

$$ (n^2 - m^2) \int_{-1}^{1} \frac{y_n(x)y_m(x)}{\sqrt{1 - x^2}} \, dx = 0$$

$$\int_{-1}^{1} \frac{y_n(x)y_m(x)}{\sqrt{1 - x^2}} \, dx = 0$$