The Laplacian in Polar Coordinates

In two-dimensional Cartesian coordinates, the Laplacian operator is expressed as

$$\nabla^2 u = u_{xx} + u_{yy}. \tag{1}$$

Making the change of variables

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

we may express the Laplacian operator in polar coordinates.

$$\begin{aligned} u_r &= u_x x_r + u_y y_r \\ &= u_x \cos\theta + u_y \sin\theta \end{aligned}$$

$$\begin{aligned} u_\theta &= u_x x_\theta + u_y y_\theta \\ &= -u_x r\sin\theta + u_y r\cos\theta \end{aligned}$$

Solving the last two equations for u_x and u_y (using Cramer's rule if necessary) yields,

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \tag{2}$$

$$u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r} \tag{3}$$

Now

$$\begin{aligned} u_{rr} &= \frac{\partial}{\partial r}\left(u_x x_r + u_y y_r\right) \\ &= \frac{\partial}{\partial r}\left(u_x \cos\theta + u_y \sin\theta\right) \\ &= \frac{\partial}{\partial r}(u_x)\cos\theta + \frac{\partial}{\partial r}(u_y)\sin\theta \\ &= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta \end{aligned}$$

and

$$\begin{aligned} u_{\theta\theta} &= \frac{\partial}{\partial\theta}\left(u_x x_\theta + u_y y_\theta\right) \\ &= \frac{\partial}{\partial\theta}\left(-u_x r\sin\theta + u_y r\cos\theta\right) \\ &= \frac{\partial}{\partial\theta}(u_x)(-r\sin\theta) + r u_x \frac{\partial}{\partial\theta}(-\sin\theta) + \frac{\partial}{\partial\theta}(u_y) r\cos\theta + r u_y \frac{\partial}{\partial\theta}\cos\theta \\ &= u_{xx} r^2 \sin^2\theta - u_{xy} r^2 \cos\theta\sin\theta - u_x r\cos\theta + u_{yy} r^2\cos^2\theta - u_{yx} r^2\cos\theta\sin\theta - u_y r\sin\theta \\ &= u_{xx} r^2 \sin^2\theta - 2u_{xy} r^2 \cos\theta\sin\theta + u_{yy} r^2 \cos^2\theta \\ &\quad - \left(u_r r\cos\theta - u_\theta \sin\theta\right)\cos\theta - \left(u_r r\sin\theta + u_\theta\cos\theta\right)\sin\theta \\ &= u_{xx} r^2 \sin^2\theta - 2u_{xy} r^2 \cos\theta\sin\theta + u_{yy} r^2\cos^2\theta - u_r r. \end{aligned}$$

After using some double-angle and half-angle formulas we can summarize the results as follows:

$$
\begin{aligned}
u_{rr} &= \frac{1}{2}(u_{xx}+u_{yy})+\frac{1}{2}\cos 2\theta(u_{xx}-u_{yy})+(\sin 2\theta)u_{xy} && (4)\\
u_{\theta\theta} &= \frac{r^2}{2}(u_{xx}+u_{yy})+\frac{r^2}{2}\cos 2\theta(u_{yy}-u_{xx})-(r^2\sin 2\theta)u_{xy}-ru_r && (5)
\end{aligned}
$$

Dividing Eq. (5) by r^2 and adding to Eq. (4) gives us

$$
\begin{aligned}
u_{rr}+\frac{1}{r^2}u_{\theta\theta} &= u_{xx}+u_{yy}-\frac{1}{r}u_r\\
\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r+u_{rr} &= u_{xx}+u_{yy}
\end{aligned}
$$

Some authors prefer to write the Laplacian in polar coordinates as

$$\nabla^2 u = \frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}\left(ru_r\right)_r . \tag{6}$$

The Laplacian in Spherical Coordinates

In three-dimensional Cartesian coordinates, the Laplacian operator is expressed as

$$\nabla^2 u = u_{xx}+u_{yy}+u_{zz}. \tag{7}$$

Making the change of variables

$$
\begin{aligned}
x &= \rho\sin\phi\cos\theta\\
y &= \rho\sin\phi\sin\theta\\
z &= \rho\cos\phi
\end{aligned}
$$

we may express the Laplacian operator in spherical coordinates.

$$
\begin{aligned}
u_\rho &= u_x x_\rho + u_y y_\rho + u_z z_\rho\\
&= u_x\sin\phi\cos\theta+u_y\sin\phi\sin\theta+u_z\cos\phi
\end{aligned}
$$

$$
\begin{aligned}
u_\phi &= u_x x_\phi + u_y y_\phi + u_z z_\phi\\
&= u_x\rho\cos\phi\cos\theta+u_y\rho\cos\phi\sin\theta-u_z\rho\sin\phi
\end{aligned}
$$

$$
\begin{aligned}
u_\theta &= u_x x_\theta + u_y y_\theta + u_z z_\theta\\
&= -u_x\rho\sin\phi\sin\theta+u_y\rho\cos\phi\cos\theta
\end{aligned}
$$

Solving the last three equations for u_x, u_y, and u_z (using Cramer's rule if necessary) yields,

$$
\begin{aligned}
u_x &= u_\rho\sin\phi\cos\theta+u_\phi\frac{\cos\phi\cos\theta}{\rho}-u_\theta\frac{\sin\theta}{\rho\sin\phi} && (8)\\
u_y &= u_\rho\sin\phi\sin\theta+u_\phi\frac{\cos\phi\sin\theta}{\rho}+u_\theta\frac{\cos\theta}{\rho\sin\phi} && (9)\\
u_z &= u_\rho\cos\phi-u_\phi\frac{\sin\phi}{\rho} && (10)
\end{aligned}
$$

Now we will take second partial derivatives.

$$\begin{aligned}
u_{\rho\rho} &= \frac{\partial}{\partial\rho}\left(u_x x_\rho + u_y y_\rho + u_z z_\rho\right) \\
&= \frac{\partial}{\partial\rho}\left(u_x \sin\phi\cos\theta + u_y \sin\phi\sin\theta + u_z\cos\phi\right) \\
&= \frac{\partial}{\partial\rho}(u_x)\sin\phi\cos\theta + \frac{\partial}{\partial\rho}(u_y)\sin\phi\sin\theta + \frac{\partial}{\partial\rho}(u_z)\cos\phi \\
&= (u_{xx}\sin\phi\cos\theta + u_{xy}\sin\phi\sin\theta + u_{xz}\cos\phi)\sin\phi\cos\theta \\
&\quad + (u_{yx}\sin\phi\cos\theta + u_{yy}\sin\phi\sin\theta + u_{yz}\cos\phi)\sin\phi\sin\theta \\
&\quad + (u_{zx}\sin\phi\cos\theta + u_{zy}\sin\phi\sin\theta + u_{zz}\cos\phi)\cos\phi \\
&= u_{xx}\sin^2\phi\cos^2\theta + 2u_{xy}\sin^2\phi\cos\theta\sin\theta + 2u_{xz}\cos\phi\sin\phi\cos\theta \\
&\quad + u_{yy}\sin^2\phi\sin^2\theta + 2u_{yz}\cos\phi\sin\phi\sin\theta + u_{zz}\cos^2\phi \\
&= u_{xx}\sin^2\phi\cos^2\theta + u_{yy}\sin^2\phi\sin^2\theta + u_{zz}\cos^2\phi + u_{xy}\sin^2\phi\sin 2\theta + u_{xz}\sin 2\phi\cos\theta \\
&\quad + u_{yz}\sin 2\phi\sin\theta
\end{aligned}$$

$$\begin{aligned}
u_{\phi\phi} &= \frac{\partial}{\partial\phi}\left(u_x x_\phi + u_y y_\phi + u_z z_\phi\right) \\
&= \frac{\partial}{\partial\phi}\left(u_x\rho\cos\phi\cos\theta + u_y\rho\cos\phi\sin\theta - u_z\rho\sin\phi\right) \\
&= \rho\frac{\partial}{\partial\phi}\left(u_x\cos\phi\cos\theta + u_y\cos\phi\sin\theta - u_z\sin\phi\right) \\
&= \rho\left[\frac{\partial}{\partial\phi}(u_x)\cos\phi\cos\theta - u_x\sin\phi\cos\theta + \frac{\partial}{\partial\phi}(u_y)\cos\phi\sin\theta - u_y\sin\phi\sin\theta - \frac{\partial}{\partial\phi}(u_z)\sin\phi\right. \\
&\quad \left. - u_z\cos\phi\right] \\
&= \rho\left[(u_{xx}\rho\cos\phi\cos\theta + u_{xy}\rho\cos\phi\sin\theta - u_{xz}\rho\sin\phi)\cos\phi\cos\theta\right. \\
&\quad - (u_\rho\sin\phi\cos\theta + u_\phi\frac{\cos\phi\cos\theta}{\rho} - u_\theta\frac{\sin\theta}{\rho\sin\phi})\sin\phi\cos\theta \\
&\quad + (u_{yx}\rho\cos\phi\cos\theta + u_{yy}\rho\cos\phi\sin\theta - u_{yz}\rho\sin\phi)\cos\phi\sin\theta \\
&\quad - (u_\rho\sin\phi\sin\theta + u_\phi\frac{\cos\phi\sin\theta}{\rho} + u_\theta\frac{\cos\theta}{\rho\sin\phi})\sin\phi\sin\theta \\
&\quad \left. - (u_{zx}\rho\cos\phi\cos\theta + u_{zy}\rho\cos\phi\sin\theta - u_{zz}\rho\sin\phi)\sin\phi - (u_\rho\cos\phi - u_\phi\frac{\sin\phi}{\rho})\cos\phi\right] \\
&= \rho^2\left[u_{xx}\cos^2\phi\cos^2\theta + u_{yy}\cos^2\phi\sin^2\theta + u_{zz}\sin^2\phi + u_{xy}\cos^2\phi\sin 2\theta - u_{xz}\sin 2\phi\cos\theta\right. \\
&\quad \left. - u_{yz}\sin 2\phi\sin\theta\right] - \rho u_\rho
\end{aligned}$$

$$
\begin{aligned}
u_{\theta\theta} &= \frac{\partial}{\partial\theta}\left(u_x x_\theta + u_y y_\theta + u_z z_\theta\right) \\
&= \frac{\partial}{\partial\theta}\left(-u_x\rho\sin\phi\sin\theta + u_y\rho\sin\phi\cos\theta\right) \\
&= \rho\frac{\partial}{\partial\theta}\left(-u_x\sin\phi\sin\theta + u_y\sin\phi\cos\theta\right) \\
&= \rho\left[-\frac{\partial}{\partial\theta}(u_x)\sin\phi\sin\theta - u_x\sin\phi\cos\theta + \frac{\partial}{\partial\theta}(u_y)\sin\phi\cos\theta - u_y\sin\phi\sin\theta\right] \\
&= -\rho\left[\frac{\partial}{\partial\theta}(u_x)\sin\phi\sin\theta + u_x\sin\phi\cos\theta - \frac{\partial}{\partial\theta}(u_y)\sin\phi\cos\theta + u_y\sin\phi\sin\theta\right] \\
&= -\rho\left[(-u_{xx}\rho\sin\phi\sin\theta + u_{xy}\rho\sin\phi\cos\theta)\sin\phi\sin\theta\right. \\
&\quad + (u_\rho\sin\phi\cos\theta + u_\phi\frac{\cos\phi\cos\theta}{\rho} - u_\theta\frac{\sin\theta}{\rho\sin\phi})\sin\phi\cos\theta \\
&\quad - (-u_{yx}\rho\sin\phi\sin\theta + u_{yy}\rho\sin\phi\cos\theta)\sin\phi\cos\theta \\
&\quad \left. + \ (u_\rho\sin\phi\sin\theta + u_\phi\frac{\cos\phi\sin\theta}{\rho} + u_\theta\frac{\cos\theta}{\rho\sin\phi})\sin\phi\sin\theta\right] \\
&= \rho^2\sin^2\phi\left[u_{xx}\sin^2\theta + u_{yy}\cos^2\theta - u_{xy}\sin 2\theta\right] - \rho u_\rho\sin^2\phi - u_\phi\cos\phi\sin\phi
\end{aligned}
$$

We can summarize the results as follows:

$$
\begin{aligned}
u_{\rho\rho} &= u_{xx}\sin^2\phi\cos^2\theta + u_{yy}\sin^2\phi\sin^2\theta + u_{zz}\cos^2\phi + u_{xy}\sin^2\phi\sin 2\theta \\
&\quad + u_{xz}\sin 2\phi\cos\theta + u_{yz}\sin 2\phi\sin\theta
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
u_{\phi\phi} &= \rho^2\left[u_{xx}\cos^2\phi\cos^2\theta + u_{yy}\cos^2\phi\sin^2\theta + u_{zz}\sin^2\phi + u_{xy}\cos^2\phi\sin 2\theta\right. \\
&\quad \left. - \ u_{xz}\sin 2\phi\cos\theta - u_{yz}\sin 2\phi\sin\theta\right] - \rho u_\rho
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
u_{\theta\theta} &= \rho^2\sin^2\phi\left[u_{xx}\sin^2\theta + u_{yy}\cos^2\theta - u_{xy}\sin 2\theta\right] \\
&\quad - \rho u_\rho\sin^2\phi - u_\phi\cos\phi\sin\phi
\end{aligned} \tag{13}
$$

Now if we add Eq. (11) to Eq. (12) divided by ρ^2 and add this sum to Eq. (13) divided by $\rho^2\sin^2\phi$ we obtain the following.

$$
\begin{aligned}
u_{\rho\rho} + \frac{u_{\phi\phi}}{\rho^2} + \frac{u_{\theta\theta}}{\rho^2\sin^2\phi} &= u_{xx}\sin^2\phi\cos^2\theta + u_{yy}\sin^2\phi\sin^2\theta + u_{zz}\cos^2\phi + u_{xy}\sin^2\phi\sin 2\theta \\
&\quad + u_{xz}\sin 2\phi\cos\theta + u_{yz}\sin 2\phi\sin\theta \\
&\quad + u_{xx}\cos^2\phi\cos^2\theta + u_{yy}\cos^2\phi\sin^2\theta + u_{zz}\sin^2\phi + u_{xy}\cos^2\phi\sin 2\theta \\
&\quad - u_{xz}\sin 2\phi\cos\theta - u_{yz}\sin 2\phi\sin\theta - \frac{1}{\rho}u_\rho \\
&\quad + u_{xx}\sin^2\theta + u_{yy}\cos^2\theta - u_{xy}\sin 2\theta - \frac{1}{\rho}u_\rho - \frac{\cos\phi}{\rho^2\sin\phi}u_\phi \\
&= -\frac{2}{\rho}u_\rho - \frac{\cos\phi}{\rho^2\sin\phi}u_\phi + u_{xx} + u_{yy} + u_{zz} \\
u_{xx} + u_{yy} + u_{zz} &= u_{\rho\rho} + \frac{1}{\rho^2}u_{\phi\phi} + \frac{1}{\rho^2\sin^2\phi}u_{\theta\theta} + \frac{2}{\rho}u_\rho + \frac{\cos\phi}{\rho^2\sin\phi}u_\phi
\end{aligned}
$$

Some authors prefer to write the Laplacian in spherical coordinates as

$$
\nabla^2 u = \frac{1}{\rho^2}\left(\rho^2 u_\rho\right)_\rho + \frac{1}{\rho^2\sin\phi}\left((\sin\phi)u_\phi\right)_\phi + \frac{1}{\rho^2\sin^2\phi}u_{\theta\theta}. \tag{14}
$$