Nondimensionalizing the Heat Equation

Students often ask about values of physical constants for the heat equation such as the thermal diffusivity, k. Most of the time we can massage the heat equation through a process known as **nondimensionalization** so that many, if not all, physical constants bearing units of measure drop out. We will illustrate this process for the heat equation below, though the process applies in general to nearly all mathematical models.

Recall the heat equation for an insulated rod of length L has the form:

$$u_t = ku_{xx}$$

where the thermal diffusivity k has units of area/time. The independent variables t and x have units of time and length respectively. Suppose we introduce new position and time variables

$$\begin{aligned} \tilde{x} &= \frac{x}{L}, \\ \tilde{t} &= \frac{kt}{L^2}. \end{aligned}$$

Notice that the units of x cancel with the units of L and thus $\tilde{x}$ is purely a number, or as we shall say "nondimensional". Likewise the $\tilde{t}$ is nondimensional.

By use of the chain rule for derivatives we see that

$$\begin{aligned} u_t &= u_{\tilde{t}} \frac{k}{L^2} \\ u_x &= u_{\tilde{x}} \frac{1}{L} \\ u_{xx} &= u_{\tilde{x}\tilde{x}} \frac{1}{L^2}. \end{aligned}$$

Substituting these expressions into the heat equation yields

$$\begin{aligned} u_{\tilde{t}} \frac{k}{L^2} &= k u_{\tilde{x}\tilde{x}} \frac{1}{L^2} \\ u_{\tilde{t}} &= u_{\tilde{x}\tilde{x}}. \end{aligned}$$

Thus after nondimensionalizing, we need not be concerned with the value of the thermal diffusivity. We can also treat the length of the rod as 1.