Orthogonality

In many of the exercises we will solve this semester we will write a function \( f(x) \) as a trigonometric series of the form

\[
 f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}.
\]

A natural question to ask is “How do we calculate the coefficients \( A_n \)?” The method is based on the orthogonality of the trigonometric functions.

Throughout this document we will make the following assumptions:

- Function \( f(x) \) is defined on the interval \([0, L]\).
- The trigonometric series converges uniformly to \( f(x) \).

To start off, consider the following integrals.

\[
 \int_{0}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx
\]

Taking the integral in Eq. (1) and applying the product-to-sum formula

\[
 \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} = \frac{1}{2} \left( \cos \left( \frac{(n - m)\pi x}{L} \right) - \cos \left( \frac{(n + m)\pi x}{L} \right) \right)
\]

we have when \( n \neq m \)

\[
 \int_{0}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \frac{1}{2} \int_{0}^{L} \cos \left( \frac{(n - m)\pi x}{L} \right) - \cos \left( \frac{(n + m)\pi x}{L} \right) \, dx
\]

\[
 = \frac{1}{2} \left( \frac{L}{(n - m)} \sin \left( \frac{(n - m)\pi x}{L} \right) - \frac{L}{(n + m)} \sin \left( \frac{(n + m)\pi x}{L} \right) \right) \bigg|_{0}^{L}
\]

\[
 = \frac{L}{2\pi} \left( \frac{1}{n + m} \sin((n + m)\pi) - \frac{1}{n - m} \sin((n - m)\pi) \right)
\]

\[
 = 0.
\]

In the case that \( n = m \) we have

\[
 \int_{0}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \int_{0}^{L} \sin^{2} \frac{n\pi x}{L} \, dx
\]

\[
 = \frac{1}{2} \int_{0}^{L} (1 - \cos \frac{2n\pi x}{L}) \, dx
\]

\[
 = \frac{L}{2} - \int_{0}^{L} \cos \frac{2n\pi x}{L} \, dx
\]

\[
 = \frac{L}{2} - \frac{L}{2n\pi} \left( \sin \frac{2n\pi x}{L} \right) \bigg|_{0}^{L}
\]

\[
 = \frac{L}{2} - \frac{L}{2n\pi} \sin(2n\pi)
\]

\[
 = \frac{L}{2}
\]
Thus to summarize
\[ \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{L}{2} & \text{if } n = m. \end{cases} \] (2)

To draw in some of the language of linear algebra, suppose we think of each \( u_n(x) = \sin \frac{n\pi x}{L} \) for \( n \in \mathbb{N} \) as a vector in the linear space of functions defined on the interval \([0, L]\) satisfying the condition that \( u_n(0) = u_n(L) = 0 \). Integration over the interval \([0, L]\) will serve as our inner product. Typically the inner product of two vectors \( u_n \) and \( u_m \) will be denoted \( \langle u_n, u_m \rangle \). Thus the inner product can be defined as
\[ \langle u_n, u_m \rangle = \int_0^L u_n(x)u_m(x) \, dx. \]

By the results derived above, when \( n \neq m \) the inner product of vectors \( u_n \) and \( u_m \) is zero or in the language of linear algebra orthogonal. The condition expressed in Eq. (2) is sometimes referred to as an orthogonality condition.

We will use the orthogonality condition to calculate the coefficients \( A_n \). Multiply both sides of the trigonometric series by \( \sin \frac{m\pi x}{L} \) and integrate from 0 to \( L \).

\[
f(x) \sin \frac{m\pi x}{L} = \sin \frac{m\pi x}{L} \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \]

\[
\int_0^L f(x) \sin \frac{m\pi x}{L} \, dx = \int_0^L \left( \sin \frac{m\pi x}{L} \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \right) \, dx \\
= \int_0^L \left( \sum_{n=1}^{\infty} A_n \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} \right) \, dx
\]

Since we have assumed the trigonometric series converges uniformly on \([0, L]\) then we may exchange the order of integration and summation. Thus we have

\[
\int_0^L \left( \sum_{n=1}^{\infty} A_n \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} \right) \, dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} \, dx \\
= A_m \frac{L}{2}
\]

by the orthogonality condition. Therefore for all \( n \in \mathbb{N} \)
\[
A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.
\]