Orthogonality

In many of the exercises we will solve this semester we will write a function $f(x)$ as a trigonometric series of the form

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}.$$

A natural question to ask is "How do we calculate the coefficients A_n?" The method is based on the orthogonality of the trigonometric functions.

Throughout this document we will make the following assumptions:

- Function $f(x)$ is defined on the interval $[0, L]$.
- The trigonometric series converges uniformly to $f(x)$.

To start off, consider the following integrals.

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx \tag{1}$$

Taking the integral in Eq. (1) and applying the product-to-sum formula

$$\sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} = \frac{1}{2}\left(\cos \frac{(n-m)\pi x}{L} - \cos \frac{(n+m)\pi x}{L}\right)$$

we have when $n \neq m$

$$\begin{aligned}
\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx &= \frac{1}{2}\int_0^L \cos \frac{(n-m)\pi x}{L} - \cos \frac{(n+m)\pi x}{L}\, dx \\
&= \frac{1}{2}\left(\frac{L}{(n-m)\pi} \sin \frac{(n-m)\pi x}{L} - \frac{L}{(n+m)\pi} \sin \frac{(n+m)\pi x}{L}\right)\Bigg|_0^L \\
&= \frac{L}{2\pi}\left(\frac{1}{n+m} \sin((n+m)\pi) - \frac{1}{n-m} \sin((n-m)\pi)\right) \\
&= 0.
\end{aligned}$$

In the case that $n = m$ we have

$$\begin{aligned}
\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx &= \int_0^L \sin^2 \frac{n\pi x}{L}\, dx \\
&= \frac{1}{2}\int_0^L (1 - \cos \frac{2n\pi x}{L})\, dx \\
&= \frac{L}{2} - \int_0^L \cos \frac{2n\pi x}{L}\, dx \\
&= \frac{L}{2} - \frac{L}{2n\pi}\left(\sin \frac{2n\pi x}{L}\Bigg|_0^L\right) \\
&= \frac{L}{2} - \frac{L}{2n\pi} \sin(2n\pi) \\
&= \frac{L}{2}
\end{aligned}$$

Thus to summarize

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{L}{2} & \text{if } n = m. \end{cases} \tag{2}$$

To draw in some of the language of linear algebra, suppose we think of each $u_n(x) = \sin\frac{n\pi x}{L}$ for $n \in \mathbb{N}$ as a vector in the linear space of functions defined on the interval $[0, L]$ satisfying the condition that $u_n(0) = u_n(L) = 0$. Integration over the interval $[0, L]$ will serve as our inner product. Typically the inner product of two vectors u_n and u_m will be denoted $\langle u_n, u_m\rangle$. Thus the inner product can be defined as

$$\langle u_n, u_m\rangle = \int_0^L u_n(x)u_m(x)\,dx.$$

By the results derived above, when $n \neq m$ the inner product of vectors u_n and u_m is zero or in the language of linear algebra **orthogonal**. The condition expressed in Eq. (2) is sometimes referred to as an **orthogonality condition**.

We will use the orthogonality condition to calculate the coefficients A_n. Multiply both sides of the trigonometric series by $\sin\frac{m\pi x}{L}$ and integrate from 0 to L.

$$\begin{aligned} f(x)\sin\frac{m\pi x}{L} &= \sin\frac{m\pi x}{L}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \\ \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx &= \int_0^L \left(\sin\frac{m\pi x}{L}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\right) dx \\ &= \int_0^L \left(\sum_{n=1}^{\infty} A_n \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\right) dx \end{aligned}$$

Since we have assumed the trigonometric series converges uniformly on $[0, L]$ then we may exchange the order of integration and summation. Thus we have

$$\begin{aligned} \int_0^L \left(\sum_{n=1}^{\infty} A_n \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\right) dx &= \sum_{n=1}^{\infty} A_n \int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx \\ &= A_m\frac{L}{2} \end{aligned}$$

by the orthogonality condition. Therefore for all $n \in \mathbb{N}$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$