# Final Examination Review

## MATH 467 *Partial Differential Equations*

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# Properties of PDEs

- Solutions
- Order of a PDE
- Linearity
- Homogeneity

# First-Order Linear PDEs

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

Characteristic curves are the solutions of $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$ and are expressed as $\psi(x,y) = k$.
Dividing by $a(x,y)$ produces first-order linear ODE:

$$\begin{aligned} u_x + \frac{b(x,y)}{a(x,y)}u_y + \frac{c(x,y)}{a(x,y)}u &= \frac{f(x,y)}{a(x,y)} \\ \frac{du}{dx} + p(x,k)u &= g(x,k). \end{aligned}$$

- General solution contains an arbitrary function of $k$.
- Side condition (if provided) can be used to determine this arbitrary function.

# Second-Order Linear PDEs

$$\begin{aligned} A(x,t)u_{tt} + B(x,t)u_{xt} + C(x,t)u_{xx} &+ D(x,t)u_t \\ + E(x,t)u_x + F(x,t)u &= G(x,t) \end{aligned}$$

Classification:

$4AC - B^2 > 0$: elliptic

$4AC - B^2 = 0$: parabolic

$4AC - B^2 < 0$: hyperbolic

# Heat Equation

Zero boundary temperatures:

$$\begin{aligned} u_t &= k\,u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= 0 \text{ for } t > 0 \\ u(L,t) &= 0 \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < L. \end{aligned}$$

Insulated boundaries:

$$\begin{aligned} u_t &= k\,u_{xx} \text{ for } 0 < x < L,\ t > 0 \\ u_x(0,t) &= 0 \text{ for } t > 0 \\ u_x(L,t) &= 0 \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < L \end{aligned}$$

# Linear Homogeneous PDEs Solutions

- Separation of variables
- Eigenfunctions
- Eigenvalues
- Fourier series

# Solutions to the Heat Equation

Zero boundary temperatures:

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 kt/L^2} \sin\left(\frac{n\pi x}{L}\right) \\
B_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx
\end{aligned}
$$

Insulated boundaries:

$$
\begin{aligned}
u(x,t) &= A_0 + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 kt/L^2} \cos\left(\frac{n\pi x}{L}\right) \\
A_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\
A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx
\end{aligned}
$$

# Fourier Series

- Periodic extensions of $f(x)$.
- Even periodic extensions of $f(x)$.
- Odd periodic extensions of $f(x)$.
- Convergence of Fourier series.
  - Piecewise continuous
  - Piecewise smooth
  - Uniform convergence
  - Weierstrass $M$-test
- Differentiation of Fourier series
- Integration of Fourier series

# Laplace's Equation

$$
\begin{aligned}
0 &= \Delta u \\
0 &= u_{xx} + u_{yy} + u_{zz} \quad \text{(rectangular coordinates)} \\
0 &= \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r + u_{rr} \quad \text{(polar coordinates)} \\
0 &= u_{\rho\rho} + \frac{2}{\rho} u_\rho + \frac{1}{\rho^2}\left( u_{\phi\phi} + \cot\phi\, u_\phi + \csc^2\phi\, u_{\theta\theta} \right) \\
&\quad \text{(spherical coordinates)}
\end{aligned}
$$

# Wave Equation

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\
u(0,t) &= 0 \text{ for } t > 0 \\
u(L,t) &= 0 \text{ for } t > 0 \\
u(x,0) &= f(x) \text{ for } 0 < x < L \\
u_t(x,0) &= g(x) \text{ for } 0 < x < L
\end{aligned}
$$

- Fourier series solution
- d'Alembert's solution

$$
u(x,t) = \frac{1}{2}\left(f(x - c\,t) + f(x + c\,t)\right) + \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} g(s)\,ds
$$

# Sturm-Liouville Theory

With $p(x) > 0$, $r(x) > 0$ for $a \leq x \leq b$,

$$\begin{aligned} \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y(x) + \lambda r(x)y(x) &= 0 \text{ for } a < x < b \\ \alpha_1 y(a) + \beta_1 \frac{dy}{dx}(a) &= 0 \\ \alpha_3 y(b) + \beta_2 \frac{dy}{dx}(b) &= 0. \end{aligned}$$

with $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$.
**Definition:** we will denote by $L[y]$ the linear operator defined as

$$L[y] = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y(x)$$

A non-trivial solution $y(x)$ to this boundary value problem is called an eigenfunction with corresponding eigenvalue $\lambda$.

# Green's Formula/Self-adjoint Operator

**Green's Formula:**

$$\int_a^b uL[v] - vL[u]\,dx = \left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_{x=a}^{x=b}$$

**Definition:** The linear operator $L[y]$ with the Sturm-Liouville boundary conditions is called **self-adjoint** if when $u$ and $v$ are any two functions satisfying the boundary conditions then

$$\int_a^b (uL[v] - vL[u])\,dx = 0.$$

# Rayleigh Quotient

$$\lambda = \frac{\left[-p(x)y(x)y'(x)\right]_{x=a}^{x=b} + \int_a^b \left[p(x)(y'(x))^2 - q(x)(y(x))^2\right]\,dx}{\int_a^b r(x)(y(x))^2\,dx}$$

# Nonhomogeneous PDEs

Heat equation:

$$\begin{aligned} u_t &= k\, u_{xx} + Q(x) \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= A \text{ for } t > 0 \\ u(L,t) &= B \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < L. \end{aligned}$$

Find an equilibrium solution $u_E(x)$ satisfying the BVP:

$$\begin{aligned} k\, u_{xx} + Q(x) &= 0 \text{ for } 0 < x < L \\ u(0) &= A \\ u(L) &= B \end{aligned}$$

Define $u(x,t) = v(x,t) + u_E(x)$ and solve for $v(x,t)$.

# Time-Dependent Sources and BCs

Heat equation:

$$\begin{aligned} u_t &= k\,u_{xx} + Q(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= A(t) \text{ for } t > 0 \\ u(L,t) &= B(t) \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < L. \end{aligned}$$

Find a reference solution $r(x,t) = \dfrac{B(t) - A(t)}{L} x + A(t)$ and define $u(x,t) = v(x,t) + r(x,t)$ and solve for $v(x,t)$ using the method of eigenfunction expansion.