1. (20 points) A tightly stretched string has its ends fixed at $x = 0$ and $x = 1$. The string is given an initial displacement $f(x) = x^3 - x$ and released with an initial velocity described by $g(x) = \sin \pi x$. The string obeys the partial differential equation:

$$u_{tt} = u_{xx} \quad \text{for } 0 < x < 1 \text{ and } t > 0.$$ 

Find the displacement of the string as a function $u(x, t)$. 

2. (20 points) The Fourier Series for $f(x) = |x|$ on the interval $[-\pi, \pi]$ is given by

$$|x| \sim \frac{\pi}{4} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)x.$$  

Let $f_N(x)$ denote the $N^{th}$ partial sum of this series. Find the mean-squared error $E$ as a function of $N$ when approximating $f(x)$ by $f_N(x)$. 
3. (20 points) Consider the Sturm-Liouville boundary value problem:

\[ \frac{d^2 \phi}{dx^2} + (\lambda - x^2) \phi(x) = 0 \quad \text{for} \quad 0 < x < 1, \]
\[ \phi(0) = 0 \]
\[ \phi(1) = 0 \]

Show that all the eigenvalues are strictly positive.
4. (7 points each) Consider the following extension of the wave equation.

\[ u_{tt} = u_{xx} - \alpha u \quad \text{for } 0 < x < 1 \text{ and } t > 0 \]
\[ u(0, t) = u(1, t) = 0 \quad \text{for } t > 0 \]
\[ u(x, 0) = f(x) \quad \text{for } 0 < x < 1 \]
\[ u_t(x, 0) = g(x) \quad \text{for } 0 < x < 1 \]

where \( \alpha > 0 \) is a positive constant.

(a) Use separation to variables to find the eigenfunctions and eigenvalues of this boundary value problem.

(b) Find the time-dependent portion of the solution.
(c) Assuming the eigenfunctions and eigenvalues are known, express the solution in terms of the time-dependent functions and the eigenfunctions.
5. Consider the operator

\[ L[u] = \frac{d^4 u}{dx^4} \]

(a) (5 points) Show that \( L[\cdot] \) is a linear operator.

(b) (5 points) Show that

\[ \int (uL[v] - vL[u]) \, dx = v'u'' - u'v'' - vu''' + uv'''. \]
(c) (10 points) Suppose that functions $u$ and $v$ both satisfy the boundary conditions

$$\begin{align*}
\phi(0) &= \phi(1) = 0 \\
\phi''(0) &= \phi'(1) = 0.
\end{align*}$$

Show that

$$\int_0^1 (uL[v] - vL[u]) \, dx = 0.$$
6. (19 points) Use the Rayleigh Quotient and an appropriate test function to estimate the smallest eigenvalue of the eigenvalue problem,

\[ \frac{d^2 \phi}{dx^2} + \lambda \phi(x) = 0 \]
\[ \phi(0) + \frac{d\phi}{dx}(0) = 0 \]
\[ \phi(1) = 0. \]