1 Bessel's Equation of Order One

Consider Bessel's equation of order one:

$$x^2y'' + xy' + (x^2-1)y = 0.$$

If we let $P(x) = x^2$, $Q(x) = x$, and $R(x) = x^2 - 1$, then we can easily see that $P(0) = 0$ (and thus $x_0 = 0$ is a singular point) and

$$\begin{aligned}
\lim_{x\to 0} \frac{xQ(x)}{P(x)} = \lim_{x\to 0} x\frac{x}{x^2} = \lim_{x\to 0} 1 = 1 &= p_0 \\
\lim_{x\to 0} \frac{x^2R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{x^2-1}{x^2} = \lim_{x\to 0} x^2 - 1 = -1 &= q_0,
\end{aligned}$$

and hence $x_0 = 0$ is a regular singular point.

The indicial equation will have the form,

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1) + r - 1 = r^2 - 1$$

Thus the indicial roots are $r_1 = 1$ and $r_2 = -1$.

Our goal is to find two linearly independent solutions to the ODE. We can always find a series solution corresponding to the larger of the two indicial roots, so we could look for a solution of the form $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$. However, since we also want a second linearly independent solution we will follow the advice of the textbook authors who suggest that we obtain a recurrence relation involving the general formula $a_n(r)$ (see page 277). Thus we will look for a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

Begin by differentiating this solution and substituting it into the differential equation. Then re-index and combine the series into a single series.

$$\begin{aligned}
x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} + (x^2-1)\sum_{n=0}^{\infty}a_nx^{n+r} &= 0 \\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + \sum_{n=0}^{\infty}(n+r)a_nx^{n+r} + \sum_{n=0}^{\infty}a_nx^{n+r+2} - \sum_{n=0}^{\infty}a_nx^{n+r} &= 0 \\
\sum_{n=0}^{\infty}[(n+r)(n+r-1) + (n+r) - 1]a_nx^{n+r} + \sum_{n=0}^{\infty}a_nx^{n+r+2} &= 0 \\
\sum_{n=0}^{\infty}[(n+r)^2 - 1]a_nx^{n+r} + \sum_{n=0}^{\infty}a_nx^{n+r+2} &= 0 \\
\sum_{n=0}^{\infty}F(r+n)a_nx^{n+r} + \sum_{n-2=0}^{\infty}a_{n-2}x^{n-2+r+2} &= 0 \\
\sum_{n=0}^{\infty}F(r+n)a_nx^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} &= 0 \\
F(r)a_0x^r + F(r+1)a_1x^{r+1} + \sum_{n=2}^{\infty}F(r+n)a_nx^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} &= 0 \\
F(r)a_0x^r + F(r+1)a_1x^{r+1} + \sum_{n=2}^{\infty}[F(r+n)a_n + a_{n-2}]x^{n+r} &= 0
\end{aligned}$$

Since $r = -1$ or $r = 1$, the first term above is zero. The second term has coefficient $a_1F(r+1) = a_1(r^2+2r)$ which must match the coefficient of x^{r+1} on the right-hand side of the equation. The quadratic part,

$r^2 + 2r \neq 0$ since $r = -1$ or $r = 1$, thus it must be the case that $a_1 = 0$. For $n \geq 2$ we have derived the following recurrence relation.

$$a_n(r) = -\frac{a_{n-2}(r)}{F(n+r)}.$$

Since $a_1(r) = 0$ then $0 = a_3(r) = a_5(r) = \cdots = a_{2n+1}(r) = \cdots$. We then can let $r = 1$, the larger of the exponents of singularity. To keep the discussion simple let $a_0 = 1$ (recall that a_0 can be any arbitrary non-zero value). Then

$$\begin{aligned} a_2(1) &= -\frac{1}{F(1+2)} = -\frac{1}{8} = -\frac{1}{2^2(1!)(2!)} \\ a_4(1) &= -\frac{a_2(1)}{F(1+4)} = \frac{1}{2^2(1!)(2!)} \cdot \frac{1}{24} = \frac{1}{2^4(2!)(3!)} \\ &\vdots \\ a_{2n}(1) &= \frac{(-1)^n}{2^{2n}(n+1)!n!} \end{aligned}$$

Thus the first solution has the form,

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n+1)!n!} x^{n+1}.$$

According to Theorem 5.7.1 the second solution will have the form

$$y_2(x) = a y_1(x) \ln x + x^{r_2} \left[1 + \sum_{n=1}^{\infty} c_n(r_2) x^n \right]$$

where the coefficients $c_n(r_2)$ satisfy the formula

$$c_n(r_2) = \frac{d}{dr} \left[(r - r_2) a_n(r) \right] \Big|_{r=r_2}$$

for $n = 1, 2, \ldots$, and

$$a = \lim_{r \to r_2} (r - r_2) a_N(r)$$

where $N = r_1 - r_2$. Let's apply these formulas in this example. First $N = 2 = 1 - (-1) = r_1 - r_2$, so

$$a = \lim_{r \to -1} (r+1) a_2(r) = \lim_{r \to -1} (r+1) \frac{-a_0}{F(r+2)} = \lim_{r \to -1} \frac{-(r+1)}{(r+2)^2 - 1} = \lim_{r \to -1} \frac{-(r+1)}{r^2 + 4r + 3} = -1. \lim_{r \to -1} \frac{-1}{r+3} = -\frac{1}{2}.$$

When n is an odd integer $a_n(r) = 0$ which implied $c_n(r_2) = 0$ when n is odd. When n is an even integer, say $n = 2m$ then

$$\begin{aligned} c_{2m}(r) &= \frac{d}{dr} \left[(r+1) \frac{-a_{2m-2}(r)}{F(r+2m)} \right] \\ &= \frac{d}{dr} \left[(r+1) \frac{a_{2m-4}(r)}{F(r+2m)F(r+2m-2)} \right] \\ &= \frac{d}{dr} \left[(r+1) \frac{-a_{2m-6}(r)}{F(r+2m)F(r+2m-2)F(r+2m-6)} \right] \\ &\vdots \\ &= \frac{d}{dr} \left[(r+1) \frac{(-1)^{m+1} a_0}{\prod_{k=1}^{m} F(r+2k)} \right] \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{m+1}\frac{d}{dr}\left[(r+1)\frac{1}{\prod_{k=1}^{m}F(r+2k)}\right]\\
&= (-1)^{m+1}\frac{d}{dr}\left[\frac{r+1}{F(r+2)\prod_{k=2}^{m}F(r+2k)}\right]\\
&= (-1)^{m+1}\frac{d}{dr}\left[\frac{r+1}{((r+2)^2-1)\prod_{k=2}^{m}F(r+2k)}\right]\\
&= (-1)^{m+1}\frac{d}{dr}\left[\frac{r+1}{(r^2+4r+3)\prod_{k=2}^{m}F(r+2k)}\right]\\
&= (-1)^{m+1}\frac{d}{dr}\left[\frac{r+1}{(r+1)(r+3)\prod_{k=2}^{m}F(r+2k)}\right]\\
&= (-1)^{m+1}\frac{d}{dr}\left[\frac{1}{(r+3)\prod_{k=2}^{m}F(r+2k)}\right].
\end{aligned}
$$

This could be very cumbersome to differentiate in general. We can calculate a few examples and look for a pattern in the derivatives.

$$c_2(r) = -\frac{d}{dr}\left(\frac{1}{r+3}\right) = \frac{1}{(r+3)^2}$$

This implies $c_2(-1) = 1/4$.

$$c_4(r) = \frac{d}{dr}\left(\frac{1}{(r+3)((r+4)^2-1)}\right) = -\frac{3r+13}{(r+3)^2(r+5)^2}$$

This implies $c_4(-1) = -5/64$. While the following is far from obvious, it is nevertheless true.

$$c_4(-1) = \frac{-5}{64} = -\frac{(1+\frac{1}{2})+1}{2^4 2!1!} = -\frac{\sum_{k=1}^{2}\frac{1}{k}+\sum_{k=1}^{1}\frac{1}{k}}{2^4 2!1!}$$

$$c_6(r) = -\frac{d}{dr}\left(\frac{1}{(r+3)((r+4)^2-1)((r+6)^2-1)}\right) = \frac{5r^2+52r+127}{(r+3)^2(r+5)^2(r+7)^2}$$

This implies

$$c_6(-1) = \frac{5}{1152} = \frac{\sum_{k=1}^{3}\frac{1}{k}+\sum_{k=1}^{2}\frac{1}{k}}{2^6 3!2!}.$$

Using proof by induction then we can see that in general

$$c_{2m}(-1) = (-1)^{m+1}\frac{\sum_{k=1}^{m}\frac{1}{k}+\sum_{k=1}^{m-1}\frac{1}{k}}{2^{2m}m!(m-1)!}$$

for $m = 1, 2, \ldots$. Hence

$$y_2(x) = -\frac{1}{2}(\ln x)y_1(x) + x^{-1}\left(1+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n-1}\frac{1}{k}}{2^{2n}n!(n-1)!}x^n\right).$$