Project Topics

During Math Awareness Week you will give a presentation during class on the work you have done on one of the topics described below. The presentation should make use of some kind of presentation software such as an electronic slideshow. The presentation is limited to ten minutes in length. Once the presentation is delivered please give me an electronic copy of it, so that I make post it on a web page of projects completed by the class.

1. Consider the boundary value problem:

\[
\begin{align*}
  u_t &= ku_{xx} \quad \text{for } x \geq 0, \ t \geq 0 \\
  u(0, t) &= \cos(\omega t) \quad \text{for } t \geq 0.
\end{align*}
\]

This describes a heat conduction problem for a semi-infinite rod whose end is subject to a time-dependent environmental temperature. A unique solution to this problem should satisfy the following two conditions.

- \( u(x, t) \to 0 \) as \( x \to \infty \), and
- \( u(x, t + 2\pi/\omega) = u(x, t) \)

(a) Find the unique solution to the boundary value problem.

(b) Show that the solution is not unique if either of the two conditions itemized above is omitted.

(c) Assuming that \( \omega = \pi/2 \) and \( k = \pi/4 \) sketch the graph of the temperature distribution in the \((x, u)\)-plane for \( t = 0, 1, 2, 3, 4 \). At what points does \( u(x, t) = 0 \)?

(d) Show that at any fixed time \( t \), the distance between consecutive local maxima, say \( x_1 \) and \( x_2 \), of \( u(x, t) \) is \( 2\pi \sqrt{2k/\omega} \). Show that the ratio \( u(x_2, t)/u(x_1, t) \) is \( e^{-2\pi} \) independent of \( k \) and \( \omega \).

2. Consider the heat equation on the sphere of radius 2.

\[
\begin{align*}
  u_t &= k(u_{xx} + u_{yy} + u_{zz}) \\
  u(x, y, z, 0) &= f(x, y, z) \\
  u(x, y, z, t) &= x^2 + 2y^2 + 3z^2 \quad \text{for } x^2 + y^2 + z^2 = 4
\end{align*}
\]

Find the steady state temperature distribution in the sphere.

3. Let \( v(\phi, \theta, t) \) be the displacement in the positive radial direction of the point \((\rho_0, \phi, \theta)\) on a vibrating sphere of equilibrium radius \( \rho_0 \). Ignoring damping, the sphere will obey the initial value problem,

\[
\begin{align*}
  v_{tt} &= a^2 \nabla^2 v - \omega^2 v \quad \text{for } 0 \leq \phi \leq \pi, \ 0 \leq \theta \leq 2\pi, \ -\infty < t < \infty \\
  v(\phi, \theta) &= f(\phi, \theta) \\
  v_t(\phi, \theta) &= g(\phi, \theta)
\end{align*}
\]

Find the formal solution to the initial value problem.

4. Consider the initial boundary value problem

\[
\begin{align*}
  u_t &= u_{xx} \quad \text{for } t > 0, \ 0 < x < 1 \\
  u(x, 0) &= f(x) \quad \text{for } 0 < x < 1 \\
  u_x(0, t) &= -u(0, t) \quad \text{for } t > 0 \\
  u_x(1, t) &= -u(1, t) \quad \text{for } t > 0
\end{align*}
\]

(a) Give a physical interpretation to this IBVP.
(b) Solve the IBVP using separation of variables.

5. The positive zeros of \( J_0(x) \) form an increasing sequence

\[ 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < \cdots. \]

In this project you will prove this fact.

(a) Plot the graph of \( J_0(x) \) for \(-1 < x < 10\).

(b) Let \( y(x) = \frac{1}{\sqrt{x}} u(x) \) and substitute this expression into Bessel's equation of order 0. Show that

\[ u'' + \left(1 + \frac{1}{4x^2}\right) u = 0. \]  

(1)

(c) Show that \( u(x) = \sqrt{x} J_0(x) \) is a solution to equation (1).

(d) Let \( v(x) = \sin x \). Show that

\[ -(u'' + u)v(x) = \frac{d}{dx} (uv' - u'v). \]

(e) From equation (1) show that

\[-(u'' + u) = \frac{u}{4x^2}. \]

(f) Show that

\[ \int_{2k\pi}^{(2k+1)\pi} \frac{u(x) \sin x}{4x^2} \, dx = -(u(2k\pi) + u((2k + 1)\pi)). \]

(g) Show that \( u(x) \) has at least one zero in the interval \([2k\pi, (2k + 1)\pi]\).

(h) Show that \( J_0(x) \) has infinitely many zeros which form an unbounded increasing sequence.

6. First order nonlinear PDEs are very important in the modeling of traffic flow, shock waves, and waves breaking the sound barrier. Consider the nonlinear problem

\[ u_t + A(u)u_x = 0 \]

\[ u(x, 0) = \phi(x) \]

(a) Find the directional derivative of \( u(x, t) \) in the direction of vector \( \langle A(u), 1 \rangle \).

(b) Show that \( u \) is constant along the curves \( x(t) \) which solve the ODE

\[ \frac{dx}{dt} = A(u(x, t)). \]

These are called the characteristic curves of the PDE.

(c) If \( x(t) \) is a characteristic curve of the PDE show that

\[ u(x(t), t) = u(x(0), 0) = \phi(x(0)). \]

(d) If \( x(t) \) is a characteristic curve of the PDE show that

\[ A(u(x(t), t)) = A(\phi(x(0))). \]

What does this imply about the rate of change of \( x \) with respect to \( t \)?
(e) Find the characteristic curves. Write the characteristic curve in implicit form, \( C(x, t) = x(0) \). The solution to the nonlinear PDE will then have the form \( u(x, t) = f(C(x, t)) \) where \( f(C(x, 0)) = \phi(x) \).

(f) Apply the solution approach outlined above to the specific example,

\[
\begin{align*}
    u_t + \ln(u) u_x &= 0 \\
    u(x, 0) &= e^x.
\end{align*}
\]

7. Consider Maxwell’s equations for the propagation of the electric field \( \mathbf{E}(x, y, z, t) \) and magnetic field \( \mathbf{B}(x, y, z, t) \) with electric charge density \( \rho(x, y, z, t) \).

\[
\begin{align*}
    \mathbf{E}_t &= c\nabla \times \mathbf{B} - 4\pi \mathbf{E} \\
    \mathbf{B}_t &= -c\nabla \times \mathbf{E} \\
    \nabla \cdot \mathbf{E} &= 4\pi \rho \\
    \nabla \cdot \mathbf{B} &= 0
\end{align*}
\]

The constant \( c \) is the speed of light and \( \sigma \) is another constant.

(a) Give a brief physical interpretation of Maxwell’s equations.

(b) Suppose we have at time \( t = 0 \) the conditions:

\[
\begin{align*}
    \mathbf{E}(x, y, z, 0) &= \mathbf{E}_0(x, y, z) \quad \text{and} \quad \mathbf{B}(x, y, z, 0) = \mathbf{B}_0(x, y, z).
\end{align*}
\]

Maxwell’s equations imply a set of partial differential equations which are satisfied by \( \mathbf{E} \) and \( \mathbf{B} \). Derive these PDEs.

8. Microscopic particles suspended in a liquid are in constant random motion called Brownian Motion. In this project we will derive a mathematical model of Brownian Motion along the one-dimensional real number line.

(a) Let \( u(x, t) \) be the probability density for the location of a particle at time \( t \). What is the probability that the particle will be located within the interval \([a, b]\) at time \( t \)?

(b) Suppose in a time interval of length \( \Delta t \) the particle will move with equal probability to the left or the right a distance \( \epsilon \equiv \epsilon(\Delta t) \). Show that

\[
u(x, t + \Delta t) = \frac{1}{2}u(x - \epsilon, t) + \frac{1}{2}u(x + \epsilon, t)
\]

(c) Use the Taylor polynomial of order 2 to show that

\[
\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \approx \frac{1}{2}u_{xx}(x, t) \frac{\epsilon^2}{\Delta t}.
\]

(d) Take the limit of both sides of equation (2) as \( \Delta t \to 0 \) and assume \( \lim_{\Delta t \to 0} \frac{\epsilon^2}{\Delta t} = \gamma \). What familiar partial differential equation results?

(e) Suppose the particle is known with absolute certainty to be at \( x = 0 \) when \( t = 0 \). What initial condition does this impose on \( u \)?

(f) Solve the initial value problem.

(g) Suppose in a time interval of length \( \Delta t \) the particle will move to the left with probability \( p \) and to the right with probability \( q \) where \( p + q = 1 \). Derive and solve the initial value problem for this situation.
9. A chain of length $L$ and density $\rho$ swings freely under its own weight from a rigid support.

(a) Show that the equation governing the motion of the chain can be written as

$$\frac{\partial^2 y}{\partial t^2} = g \frac{\partial}{\partial x} \left( x \frac{\partial y}{\partial x} \right)$$

(b) Let $y(x, t) = f(x)e^{-\omega t}$ and introduce the dimensionless length variable $w^2 = x/L$. Show that as a function of $w$, $f$ satisfies

$$\frac{d}{dw} \left( w^2 \frac{df}{dw} \right) + \frac{4\omega^2 L}{g}$$

which has solution $J_0(2\omega \sqrt{L/g})$

(c) Show that the lowest frequency of vibration $\nu_1 = \omega_1/(2\pi)$ is approximately equal to $1.203\nu_p$ where $\nu_p$ is the frequency of a pendulum of length $L$. Note that for a rigid bar the oscillation frequency is $\frac{\sqrt{2}}{2} \nu_p$.

10. The initial boundary value problem for the deflection $u(x, t)$ of a uniform beam of length $L$ hinged at its ends is

$$u_{tt} + a^4 u_{xxxx} = 0 \quad \text{for} \quad 0 < x < L, \; t > 0$$

$$u(x, 0) = x(L - x) \quad \text{for} \quad 0 < x < L$$

$$u(0, t) = 0 \quad \text{for} \quad 0 < t$$

$$u(L, t) = 0 \quad \text{for} \quad 0 < t$$

$$u_{xx}(0, t) = 0 \quad \text{for} \quad 0 < t$$

$$u_{xx}(L, t) = 0 \quad \text{for} \quad 0 < t$$

(a) Given a physical interpretation of the boundary conditions.

(b) Use separation of variables to solve the IBVP.

11. A solid sphere has radius $R$ and diffusivity $k$ and its initial temperature distribution is given by $f(\rho) = \rho(1 - \rho)$. Heat radiates from the surface of the sphere via the law

$$\frac{\partial u}{\partial \rho}(R, t) = -hu(R, t)$$

where $h$ is a positive constant and $Rh < 1$. Find the temperature $u(\rho, t)$ of the sphere. Plot an approximation to the solution when $R = \pi$ inches, $k = 5.58$ inches-squared per hour, and $h = 0.1$ per inch.

12. Consider a sphere of radius 1. Suppose the surface of the top half of the sphere is kept at a constant 100 degrees while the surface of the bottom of the sphere is kept at a constant 0 degrees.

(a) Write down the boundary value problem in spherical coordinates for the situation described.

(b) Find the steady state temperature distribution for the sphere.

(c) Graph the steady state temperature distribution for the slice of the sphere which passes through its north and south poles and which is perpendicular to its equator.

13. Determine the steady state temperature distribution inside a solid hemisphere $H = \{(\rho, \phi, \theta) \mid 0 \leq \rho \leq 1, \; 0 \leq \phi \leq \frac{\pi}{2}, \; 0 \leq \theta \leq 2\pi\}$, when the base at $\phi = \frac{\pi}{2}$ is insulated but the temperature on the curved surface is $f(\phi) = 1 + \cos(2\phi)$.

14. A solid cylinder of radius 1 and height 4 has its base and round side kept at temperature 0 while the top end is kept at temperature $f(r) = 1 - r^2$. 

4
(a) State the boundary value problem for the steady-state temperature at any point in the cylinder.
(b) Find the steady-state temperature at any point in the cylinder.
(c) Plot the steady-state temperature for several slices of the cylinder parallel to its base.

15. Consider heat conduction in a cylindrical (hollow) pipe of length \( L \) and radius \( a \). Let the initial temperature distribution on the pipe be given by the function \( f(\theta, z) = 1 + \frac{z}{L} \cos \theta \). Suppose the ends of the pipe are insulated. Find the solution to the heat equation in this situation.

16. A semi-circular plate of radius 1 has its boundary kept at temperature 0 and its faces insulated. The initial temperature in the plate is given by \( f(r, \theta) = 100r^2 \cos \theta \).

(a) State the initial boundary value problem describing this situation.
(b) Find the temperature \( u(r, \theta, t) \) inside the plate at any time.
(c) Plot the temperature inside the plate for several different times.

17. A spherical shell has an inner radius of 1 and an outer radius of 2. The temperatures of the inner and outer surfaces are respectively \( f_i(\theta) = 100 - 20 \cos \theta \) and \( f_o(\theta) = 50 + 30 \cos \theta \).

(a) State the boundary value problem for the steady-state temperature within the shell.
(b) Find the steady-state temperature inside the shell.
(c) Plot the steady-state temperature inside the shell for several slices parallel to the shell’s equator.

18. A cylinder of radius 1 and height 1 has its base kept at temperature 0 and its top and round side kept at temperature 100.

(a) State the boundary value problem for the steady-state temperature at any point in the cylinder.
(b) Find the steady-state temperature at any point in the cylinder.
(c) Plot the steady-state temperature for several slices of the cylinder parallel to its base and for the axial slice through the cylinder.

19. A sphere of radius 1 has an initial temperature \( f(\rho) = \cos(\pi \rho/2) \). The surface of the sphere is kept at temperature 0.

(a) State the initial boundary value problem for the temperature at any point in the sphere.
(b) Find the temperature at any point in the sphere.
(c) Plot the temperature in the equatorial slice of the sphere for several positive values of \( t \).

20. Consider an infinitely long string made of two different materials joined at \( x = 0 \). For \( x < 0 \) the density of the string is \( \rho_1 \) while for \( x > 0 \) the density of the string is \( \rho_2 \).

(a) State the partial differential equation which the vibration of the string will obey.
(b) Suppose the initial displacement of the string is given by \( u(x, 0) = f(x) \) where \( f(x) = 0 \) for \( x > 0 \). Find the displacement of the string \( u(x, t) \) for \( t > 0 \).
(c) What effect does the material change have on the solution to the wave equation?

21. The Schrödinger equation for the hydrogen atom is

\[
E \Psi = - \left( \frac{\hbar^2}{2m} \Delta + \frac{Ze^2}{4\pi \epsilon_0 \rho} \right) \Psi
\]

where \( \Psi = \Phi(\rho, \theta, \phi) \) and \( \Delta \) represents the Laplacian operator in spherical coordinates.
(a) Assuming the solution to the equation is \( \Psi(\rho, \theta, \phi) = R(\rho)Y(\theta, \phi) \) show that

\[
\frac{\hbar^2}{2m} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right] = \lambda Y
\]

and

\[
\frac{\hbar^2}{2m} \left[ \frac{d}{d\rho} \left( \rho^2 R' \right) \right] + \frac{Ze^2 \rho R}{4\pi\varepsilon_0} + \rho^2 RE = -\lambda R
\]

for some constant \( \lambda \).

(b) Assume that \( Y(\phi, \theta) = P(\phi)T(\theta) \) and that \( P(\phi) \) is periodic. Show that

\[
P(\phi) = e^{\pm in\phi}, \quad \text{with } n \in \mathbb{N} \cup \{0\}, \text{ and}
\]

\[
0 = \frac{1}{T} \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \lambda \sin^2 \theta - n^2
\]

(c) Assume that \( \xi = \cos \theta \) and show that

\[
\frac{d}{d\xi} \left[ (1 - \xi^2) \frac{dT}{d\xi} \right] - \frac{n^2}{1 - \xi^2} T + \lambda T = 0 \quad \text{with } \lambda = l(l + 1), \text{ and}
\]

\[
Y_l^n(\phi, \theta) = P_l^n(\cos \theta)e^{in\phi}
\]

where \( P_l^n \) is the associated Legendre function.