Problem Description

Consider the heat equation on the sphere of radius 2.

\[
\begin{align*}
  u_t &= k(u_{xx} + u_{yy} + u_{zz}) \\
  u(x, y, z, 0) &= f(x, y, z) \\
  u(x, y, z, t) &= x^2 + 2y^2 + 3z^2 \quad \text{for } x^2 + y^2 + z^2 = 4, \, t > 0
\end{align*}
\]

Find the steady state temperature distribution in the sphere.

The temperature distribution on the surface of the sphere is shown in the plot below.
Problem Solution

At steady state, \( u_t = 0 \) and thus we must solve Laplace’s equation on the sphere.

\[
0 = u_{xx} + u_{yy} + u_{zz}
\]
\[
u(x, y, z) = x^2 + 2y^2 + 3z^2 \quad \text{for} \quad x^2 + y^2 + z^2 = 4
\]

Since the physical domain of the problem is a sphere, it will be to our advantage to convert the Laplacian and the boundary conditions to spherical coordinates. We will adopt the physical rather than mathematical convention of labeling the azimuthal angle \( \phi \) and the latitudinal angle \( \theta \). Thus to convert from Cartesian to spherical coordinates we will make use of the following equations.

\[
x = \rho \sin \theta \cos \phi \nn\]
\[
y = \rho \sin \theta \sin \phi \nn\]
\[
z = \rho \cos \theta
\]

In spherical coordinates the Laplacian operator \( u_{xx} + u_{yy} + u_{zz} = \Delta u \) has the form

\[
\Delta u = \frac{1}{\rho^2 \frac{\partial}{\partial \rho}} \left( \rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \theta \frac{\partial}{\partial \theta}} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2 \theta \frac{\partial^2 u}{\partial \phi^2}}
\]

where the coordinate \( \rho \) is the distance from the origin. The details of the conversion may be found in Appendix A.

Since Laplace’s equations is linear and homogeneous, we may attempt to solve it using the method of separation of variables. To begin we will assume that \( u(\rho, \theta, \phi) = R(\rho)T(\theta)P(\phi) \) and substitute this expression in Laplace’s equation. This produces

\[
\frac{1}{\rho^2 \frac{\partial}{\partial \rho}} \left( \rho^2 R' \right) TP + \frac{1}{\rho^2 \sin \theta \frac{\partial}{\partial \theta}} \left( \sin \theta T' \right) RP + \frac{1}{\rho^2 \sin^2 \theta} \frac{R''}{P} = 0
\]

where the “prime” notation denotes the appropriate ordinary derivative. Multiplying both sides of this equation by \( \frac{\rho^2 \sin^2 \theta}{RTP} \) yields the equation

\[
\frac{\sin^2 \theta}{R} \frac{\partial}{\partial \rho} \left( \rho^2 R' \right) + \frac{\sin \theta}{T} \frac{\partial}{\partial \theta} \left( \sin \theta T' \right) + \frac{P''}{P} = 0
\]

Moving the functions depending only on \( \phi \) to the right-hand side of the equations allows us to separate the \( \phi \) and \( (\rho, \theta) \) variables.

\[
\frac{\sin^2 \theta}{R} \frac{\partial}{\partial \rho} \left( \rho^2 R' \right) + \frac{\sin \theta}{T} \frac{\partial}{\partial \theta} \left( \sin \theta T' \right) = -\frac{P''}{P}
\]

Since the left-hand side of the equation depends only on \( (\rho, \theta) \) and the right-hand side depends only on \( \phi \), both sides of the equation must equal a constant.
\( \phi \)-Dependent Factor of \( u \)

Since the solution to Laplace’s equation must be \( 2\pi \)-periodic in \( \phi \) the constant must be of the form \( n^2 \) where \( n \in \mathbb{N} \cup \{0\} \). Hence

\[
\frac{P''}{P} = n^2 \\
P'' + n^2 P = 0 \\
P_n(\phi) = A_n \cos n\phi + B_n \sin n\phi.
\]

See [1, Chap. 3] for the classical method of solving a second-order, linear, homogeneous ordinary differential equation of the type shown above.

Now we turn to the task of solving for the remaining two factors of the solution \( u \) to Laplace’s equation.

\( \rho \)-Dependent Factor of \( u \)

The partial solution \( P_n(\phi) \) found in the previous section imposes the following constraint on the solutions for \( \rho \) and \( \theta \).

\[
\frac{\sin^2 \theta}{R} \frac{\partial}{\partial \rho} \left( \rho^2 R' \right) + \frac{\sin \theta}{T \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta T' \right) = n^2,
\]

where \( n \in \mathbb{N} \cup \{0\} \). As a next step we divide both sides of the equation by \( \sin^2 \theta \) to obtain

\[
\frac{1}{R} \frac{\partial}{\partial \rho} \left( \rho^2 R' \right) + \frac{1}{T \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta T' \right) = \frac{n^2}{\sin^2 \theta} \\
\frac{1}{R} \frac{\partial}{\partial \rho} \left( \rho^2 R' \right) = \frac{n^2}{\sin^2 \theta} - \frac{1}{T \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta T' \right).
\]

We see that the left-hand side of the equation depends only on \( \rho \) while the right-hand side depends only on \( \theta \). Thus both sides are equal to a constant. If the constant has the form \( m(m+1) \) where \( m \in \mathbb{N} \cup \{0\} \) then the following ordinary differential equation is implied by the left-hand side of the last equation.

\[
\frac{1}{R} \frac{\partial}{\partial \rho} \left( \rho^2 R' \right) = m(m+1) \\
\rho^2 R'' + 2\rho R' - m(m+1)R = 0
\]

The latter equation is known as Euler’s Equation (see [1, Sec. 5.5]). The solution to this equation has the form

\[
R(\rho) = C\rho^m + D\rho^{-m-1}.
\]

Since the solution to Laplace’s equation must be bounded at the origin (where \( \rho = 0 \)) we must assume \( D = 0 \) and thus the \( \rho \)-dependent factor of the solution has the form \( R_m(\rho) = \rho^m \) with \( m \in \mathbb{N} \cup \{0\} \).

Now we proceed to the task of finding the \( \theta \)-dependent factor of the solution to Laplace’s equation.
\(\theta\)-Dependent Factor of \(u\)

The solutions found for \(R(\rho)\) and \(P(\phi)\) have imposed two constraints on the differential equation for \(T(\theta)\). It now has the form

\[
\frac{n^2}{\sin^2 \theta} - \frac{1}{T \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta T' \right) = m(m + 1),
\]

which is equivalent to the ordinary differential equation

\[
T'' + \frac{\cos \theta}{\sin \theta} T' + \left( m(m + 1) - \frac{n^2}{\sin^2 \theta} \right) = 0.
\]

We can put this equation in a more convenient form for solving by making the change of variable \(w = \cos \theta\). In this case,

\[
\frac{dT}{d\theta} = -\sin \theta \frac{dT}{dw} \quad \text{and} \quad \frac{d^2 T}{d\theta^2} = -\cos \theta \frac{dT}{dw} + \sin^2 \theta \frac{d^2 T}{dw^2}.
\]

Substituting these expressions in the previous ordinary differential equation gives us

\[
\sin^2 \theta \frac{d^2 T}{dw^2} - 2 \cos \theta \frac{dT}{dw} + \left( m(m + 1) - \frac{n^2}{\sin^2 \theta} \right) T = 0
\]

\[
(1 - \cos^2 \theta) \frac{d^2 T}{dw^2} - 2 \cos \theta \frac{dT}{dw} + \left( m(m + 1) - \frac{n^2}{1 - \cos^2 \theta} \right) T = 0
\]

\[
(1 - w^2) \frac{d^2 T}{dw^2} - 2w \frac{dT}{dw} + \left( m(m + 1) - \frac{n^2}{1 - w^2} \right) T = 0.
\]

This last ordinary differential equation is known as the associated Legendre differential equation (see [3]). The solutions will be denoted by the functions \(T^m_n(w) = T^m_n(\cos \theta)\) with \(m \in \mathbb{N} \cup \{0\}\) and \(0 \leq n \leq m\) (see [4]). Now we may summarize the product solution.

**General Solution**

Combining the three factors of the product solution we have

\[
u^m_n(\rho, \theta, \phi) = \rho^m T^m_n(\cos \theta)(A^m_n \cos n\phi + B^m_n \sin n\phi)
\]

for \(m \in \mathbb{N} \cup \{0\}\) and \(0 \leq n \leq m\). Thus the series solution of Laplace’s equation on the sphere can be written as

\[
u(\rho, \theta, \phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \rho^m T^m_n(\cos \theta)(A^m_n \cos n\phi + B^m_n \sin n\phi).
\]

The next step is to choose the coefficients \(A^m_n\) and \(B^m_n\) so that the boundary conditions can be satisfied.
Boundary Conditions

The boundary of the sphere is kept at a temperature distribution described by

$$u(x, y, z) = x^2 + 2y^2 + 3z^2 \quad \text{for } x^2 + y^2 + z^2 = 4.$$  

In spherical coordinates these boundary conditions are equivalent to

$$u(2, \theta, \phi) = 6 + 6\cos^2 \theta - 2(1 - \cos^2 \theta)\cos 2\phi$$  

for $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The details of this coordinate conversion are found in Appendix B.

Rather than try to use the orthogonality property of the eigenfunctions, we will use the fact that the boundary condition is a finite sum of eigenfunctions and thus we merely need match the coefficients at the boundary. The first several associated Legendre functions are listed in the matrix below.

<table>
<thead>
<tr>
<th>( T_0^0(w) )</th>
<th>( T_1^0(w) )</th>
<th>( T_1^1(w) )</th>
<th>( T_2^0(w) )</th>
<th>( T_1^2(w) )</th>
<th>( T_2^2(w) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( w )</td>
<td>( -\sqrt{1-w^2} )</td>
<td>( \frac{1}{2}(3w^2-1) )</td>
<td>( -3w\sqrt{1-w^2} )</td>
<td>( 3(1-w^2) )</td>
</tr>
</tbody>
</table>

Thus by writing out the first several terms of the solution evaluated when \( \rho = 2 \) we get

$$u(2, \theta, \phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} 2^m T_m^n(\cos \theta)(A_m^n \cos n\phi + B_m^n \sin n\phi)$$

$$= A_0^0 T_0^0(\cos \theta) + 2A_1^0 T_1^0(\cos \theta) + 2T_1^1(\cos \theta)(A_1^1 \cos \phi + B_1^1 \sin \phi)$$

$$+ 4A_2^0 T_2^0(\cos \theta) + 4T_1^1(\cos \theta)(A_2^1 \cos \phi + B_2^1 \sin \phi) + 4T_2^2(\cos \theta)(A_2^2 \cos 2\phi + B_2^2 \sin 2\phi)$$

$$+ \cdots$$

We can see by the absence of \( \sin n\phi \) terms in the spherical coordinate form of the boundary conditions that \( B_j^i = 0 \) for all \( i \) and \( j \). Absent as well are any powers of \( \cos \theta \) larger than 2, thus \( A_{m}^{n} = 0 \) for \( n > 2 \). Thus we have

$$u(2, \theta, \phi) = A_0^0 + 2A_1^0 \cos \theta - 2A_1^1 \sin \theta \cos \phi + 2A_2^0(3\cos^2 \theta - 1) - 12A_2^1 \cos \theta \sin \theta \cos \phi$$

$$+ 12A_2^2(1 - \cos^2 \theta) \cos 2\phi.$$  

Simplifying one step at a time we can set \( A_0^0 = A_1^1 = A_2^2 = 0 \). This will leave,

$$u(2, \theta, \phi) = A_0^0 + 2A_1^0(3\cos^2 \theta - 1) + 12A_2^0(1 - \cos^2 \theta) \cos 2\phi$$

$$= (A_0^0 - 2A_1^0) + 6A_0^0 \cos^2 \theta + 12A_0^0(1 - \cos^2 \theta) \cos 2\phi$$

and hence if we choose \( A_0^0 = 1, A_0^0 = 8, \text{ and } A_2^0 = -\frac{1}{6} \) we have satisfied the boundary conditions. Therefore the solution to Laplace’s equation on the sphere may be written as

$$u(\rho, \theta, \phi) = 8 + \frac{1}{2}\rho^2(3\cos^2 \theta - 1) - \frac{1}{2}\rho^2(1 - \cos^2 \theta) \cos 2\phi.$$  

5
Plots of Solution

The following sequence of frames illustrates the $\rho$-dependency of the solution for $0 \leq \rho \leq 2$ in steps of size $\Delta \rho = \frac{2}{7}$.
The following sequence of frames illustrates the $\phi$-dependency of the solution for $0 \leq \phi \leq \frac{7\pi}{4}$ in steps of size $\Delta \phi = \frac{\pi}{4}$.
The following sequence of frames illustrates the $\theta$-dependency of the solution for $0 \leq \theta \leq \frac{7\pi}{8}$ in steps of size $\Delta \theta = \frac{\pi}{8}$.
A Spherical Laplacian

The Laplacian operator in spherical coordinates can be derived by use of the chain rule for multivariable functions (see [2, Sec. 12.5]). Suppose that \( u \equiv u(\rho, \theta, \phi) \) then the first partial derivatives of \( u \) are

\[
\begin{align*}
\rho_u &= u_{\rho} = u_{xx} \rho + u_{yy} \rho + u_{zz} \rho \\
\theta_u &= u_{\theta} = u_{x} \cos \theta + u_{y} \sin \theta \\
\phi_u &= u_{\phi} = u_{x} \sin \theta \cos \phi + u_{y} \sin \theta \sin \phi + u_{z} \cos \theta
\end{align*}
\]

In a similar fashion we may find the three (out of the nine total) second partial derivatives.

\[
\begin{align*}
\rho_{\rho u} &= u_{xx} \sin^2 \theta \cos^2 \phi + 2u_{xy} \sin^2 \theta \cos \phi \sin \phi + 2u_{xz} \cos \theta \sin \phi + u_{yy} \sin^2 \theta \sin^2 \phi \\
&+ 2u_{yz} \cos \theta \sin \phi + u_{zz} \cos^2 \theta \\
\theta_{\theta u} &= u_{xx} \rho^2 \cos^2 \theta \cos^2 \phi + 2u_{xy} \rho^2 \cos^2 \theta \cos \phi \sin \phi - 2u_{xz} \rho^2 \cos \theta \cos \phi + u_{yy} \rho^2 \cos^2 \theta \sin^2 \phi \\
&- 2u_{yz} \rho^2 \cos \theta \sin \phi + u_{zz} \rho^2 \sin^2 \theta - u_{xx} \rho \sin \phi + u_{yy} \rho \cos \phi + u_{zz} \rho \sin \phi \\
\phi_{\phi u} &= u_{xx} \rho^2 \sin^2 \theta \sin^2 \phi - 2u_{xy} \rho^2 \sin^2 \theta \cos \phi \sin \phi + u_{yy} \rho^2 \sin^2 \theta \cos^2 \phi - u_{xz} \rho \sin \phi + u_{yz} \rho \cos \phi
\end{align*}
\]

Now we can verify that

\[
u_{xx} + u_{yy} + u_{zz} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}.
\]
We start with the right-hand side.

\[
\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = u_{\rho \rho} + \frac{2}{\rho} u_{\rho \theta} + \frac{1}{\rho^2 \sin \theta} u_{\theta \theta} + \frac{1}{\rho^2 \sin^2 \theta} \rho^2 u_{\phi \phi} \\
= u_{xx} \sin^2 \theta \cos^2 \phi + 2u_{xy} \sin^2 \theta \cos \phi \sin \phi + 2u_{xz} \cos \theta \sin \theta \cos \phi + u_{yy} \sin^2 \theta \sin^2 \phi \\
\quad + 2u_{yz} \cos \theta \sin \theta \sin \phi + u_{zz} \cos^2 \theta + \frac{2}{\rho} (u_x \sin \theta \cos \phi + u_y \sin \theta \sin \phi + u_z \cos \theta) \\
\quad + u_{xx} \cos^2 \theta \cos^2 \phi + 2u_{xy} \cos^2 \theta \cos \phi \sin \phi - 2u_{xz} \cos \theta \sin \theta \cos \phi + u_{yy} \cos^2 \theta \sin^2 \phi \\
\quad - 2u_{yz} \cos \theta \sin \theta \sin \phi + u_{zz} \sin^2 \theta - \frac{1}{\rho} (u_x \sin \theta \cos \phi + u_y \sin \theta \sin \phi + u_z \cos \theta) \\
\quad + \frac{\cos \theta}{\rho^2 \sin \theta} (u_x \rho \cos \theta \cos \phi + u_y \rho \cos \theta \sin \phi - u_z \rho \sin \theta) + u_{xx} \sin^2 \phi \\
- 2u_{xy} \cos \phi \sin \phi + u_{yy} \cos^2 \phi - \frac{1}{\rho \sin \theta} (u_x \cos \phi + u_y \sin \phi) \\
\quad + u_{xx} \left( [\sin^2 \theta + \cos^2 \theta] \cos^2 \phi + \sin^2 \phi \right) + u_{yy} \left( [\sin^2 \theta + \cos^2 \theta] \sin^2 \phi + \cos^2 \phi \right) + u_{zz} \left( \cos^2 \theta + \sin^2 \theta \right) \\
\quad + 2u_{xy} \left( [\sin^2 \theta + \cos^2 \theta] \cos \phi \sin \phi - \cos \phi \sin \phi \right) + 2u_{xz} \left( \cos \theta \sin \theta \cos \phi - \cos \theta \sin \phi \cos \phi \right) \\
\quad + 2u_{yz} \left( \cos \theta \sin \theta \sin \phi - \cos \theta \sin \phi \sin \phi \right) + \frac{\cos^2 \theta}{\rho \sin \theta} (u_x \cos \phi + u_y \sin \phi) - \frac{1}{\rho} u_z \cos \theta \\
\quad + \frac{1}{\rho} (u_x \sin \theta \cos \phi + u_y \sin \theta \sin \phi + u_z \cos \theta) - \frac{1}{\rho \sin \theta} (u_x \cos \phi + u_y \sin \phi) \\
= u_{xx} + u_{yy} + u_{zz} + \frac{1 - \sin^2 \theta}{\rho \sin \theta} (u_x \cos \phi + u_y \sin \phi) + \frac{1}{\rho} (u_x \sin \theta \cos \phi + u_y \sin \theta \sin \phi) \\
- \frac{1}{\rho \sin \theta} (u_x \cos \phi + u_y \sin \phi) \\
= u_{xx} + u_{yy} + u_{zz} - \frac{\sin \theta}{\rho} (u_x \cos \phi + u_y \sin \phi) + \frac{1}{\rho} (u_x \sin \theta \cos \phi + u_y \sin \theta \sin \phi) \\
= u_{xx} + u_{yy} + u_{zz}
\]
B Spherical Boundary Conditions

\[ u(x, y, z) = x^2 + 2y^2 + 3z^2 \quad \text{for} \quad x^2 + y^2 + z^2 = 4 \]
\[ u(2, \theta, \phi) = (2 \sin \theta \cos \phi)^2 + 2(2 \sin \theta \sin \phi)^2 + 3(2 \cos \theta)^2 \]
\[ = 4 \left( \sin^2 \theta \cos^2 \phi + 2 \sin^2 \theta \sin^2 \phi + 3 \cos^2 \theta \right) \]
\[ = 4 \left( [1 - \cos^2 \theta] \cos^2 \phi + 2[1 - \cos^2 \theta] \sin^2 \phi + 3 \cos^2 \theta \right) \]
\[ = 4 \left( [3 - \cos^2 \phi - 2 \sin^2 \phi] \cos^2 \theta + 2 \sin^2 \phi + \cos^2 \phi \right) \]
\[ = 4 \left( [2 - \sin^2 \phi] \cos^2 \theta + 1 + \sin^2 \phi \right) \]
\[ = 4 \left( [2 - \frac{1 - \cos 2\phi}{2}] \cos^2 \theta + 1 + \frac{1 - \cos 2\phi}{2} \right) \]
\[ = 2(3 + \cos 2\phi) \cos^2 \theta + 2(3 - \cos 2\phi) \]
\[ = 6 + 6 \cos^2 \theta - 2(1 - \cos^2 \theta) \cos 2\phi \]