Early Exercise

Since American style options give the holder the same rights as European style options \textit{plus} the possibility of early exercise we know that

$$C^e \leq C^a \quad \text{and} \quad P^e \leq P^a.$$
Early Exercise

An American option (a call, for instance) may have a positive payoff even when the corresponding European call has zero payoff.
Trade-offs of Early Exercise

Consider an American-style call option on a dividend-paying stock. If the option is exercised early you,

- own the stock and are entitled to receive any dividends paid after exercise,
Trade-offs of Early Exercise

Consider an American-style call option on a dividend-paying stock. If the option is exercised early you,

- own the stock and are entitled to receive any dividends paid after exercise,
- pay the strike price $K$ early foregoing the interest
  \[ K \left( e^{r(T-t)} - 1 \right) \]
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Consider an American-style call option on a dividend-paying stock. If the option is exercised early you,

- own the stock and are entitled to receive any dividends paid after exercise,
- pay the strike price $K$ early foregoing the interest $K \left(e^{r(T-t)} - 1\right)$ you could have earned on it, and
- lose the insurance provided by the call in case $S(T) < K$. 
Recall: European options obey the **Put-Call Parity Formula**:

\[ P^e + S = C^e + Ke^{-rT} \]
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American options do not satisfy a parity formula, but some inequalities must be satisfied.
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American options do not satisfy a parity formula, but some inequalities must be satisfied.

Theorem

Suppose the current value of a security is \( S \), the risk-free interest rate is \( r \), and \( C^a \) and \( P^a \) are the values of an American call and put respectively on the security with strike price \( K \) and expiry \( T > 0 \). Then

\[ C^a + K \geq S + P^a \]
Proof

Assume to the contrary that $C^a + K < S + P^a$.

- Sell the security, sell the put, and buy the call. This produces a cash flow of $S + P^a - C^a$.
- Invest this amount at the risk-free rate,
- If the owner of the American put chooses to exercise it at time $0 \leq t \leq T$, the call option can be exercised to purchase the security for $K$.
- The net balance of the investment is
  \[
  (S + P^a - C^a)e^{rt} - K > Ke^{rt} - K \geq 0.
  \]
- If the American put expires out of the money, exercise the call to close the short position in the security at time $T$.
  The net balance of the investment is
  \[
  (S + P^a - C^a)e^{rT} - K > Ke^{rT} - K > 0.
  \]

Thus the investor receives a non-negative profit in either case, violating the principle of no arbitrage.
Another Inequality

Theorem
Suppose the current value of a security is $S$, the risk-free interest rate is $r$, and $C^a$ and $P^a$ are the values of an American call and put respectively on the security with strike price $K$ and expiry $T > 0$. Then

$$S + P^a \geq C^a + Ke^{-rT}$$
Proof

Suppose $S + P_a < C^a + Ke^{-rT}$.

- Sell an American call and buy the security and the American put. Thus $C^a - S - P_a$ is borrowed at $t = 0$.
- If the owner of the call decides to exercise it at any time $0 \leq t \leq T$, sell the security for the strike price $K$ by exercising the put. The amount of loan to be repaid is $(C^a - S - P_a)e^{rt}$ and

$$
(C^a - S - P_a)e^{rt} + K = (C^a + Ke^{-rt} - S - P_a)e^{rt} \\
\geq (C^a + Ke^{-rT} - S - P_a)e^{rt}
$$

since $r > 0$. By assumption $S + P_a < C^a + Ke^{-rT}$, so the last expression above is positive.
Combining the results of the last two theorems we have the following inequality.

\[ S - K \leq C^a - P^a \leq S - Ke^{-rT} \]
Example

The price of a security is currently $36, the risk-free interest rate is 5.5% compounded continuously, and the strike price of a six-month American call option worth $2.03 is $37. The range of no arbitrage values of a six-month American put on the same security with the same strike price is

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\[ 36 - 37 \leq 2.03 - P^a \leq 36 - 37 e^{-0.055(6/12)} \]
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\[ 36 - 37 \leq 2.03 - P^a \leq 36 - 37e^{-0.055(6/12)} \]

\[ 2.03 \leq P^a \leq 3.03 \]
A Surprising Equality

We know $C^a \geq C^e$, but in fact:

Theorem

If $C^a$ and $C^e$ are the values of American and European call options respectively on the same underlying non-dividend-paying security with identical strike prices and expiry times, then $C^a = C^e$.

Remark: American calls on non-dividend-paying stocks are not exercised early.
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*If $C^a$ and $C^e$ are the values of American and European call options respectively on the same underlying non-dividend-paying security with identical strike prices and expiry times, then*

$$C^a = C^e.$$  

**Remark:** American calls on non-dividend-paying stocks are not exercised early.
Proof

Suppose that $C^a > C^e$.

- Sell the American call and buy a European call with the same strike price $K$, expiry date $T$, and underlying security. The net cash flow $C^a - C^e > 0$ would be invested at the risk-free rate $r$.

- If the owner of the American call chooses to exercise the option at some time $t \leq T$, sell short a share of the security for amount $K$ and add the proceeds to the amount invested at the risk-free rate.

- At time $T$ close out the short position in the security by exercising the European option. The amount due is

$$
(C^a - C^e)e^{rT} + K(e^{r(T-t)} - 1) > 0.
$$

- If the American option is not exercised, the European option can be allowed to expire and the amount due is

$$
(C^a - C^e)e^{rT} > 0.
$$
Theorem

For a non-dividend-paying stock whose current price is $S$ and for which the American put with a strike price of $K$ and expiry $T$ has a value of $P^a$, satisfies the inequality

$$(K - S)^+ \leq P^a < K.$$
Proof

- Suppose $P^a < K - S$.
  - Buy the put and the stock (cost $P^a + S$).
  - Immediately exercise the put and sell the stock for $K$.
  - Net transaction $K - P^a - S > 0$ (arbitrage).

- Suppose $P^a > K - S$.
  - Sell the put and invest proceeds at risk-free rate $r$.
    Amount due at time $t$ is $P^a e^{rt}$.
  - If the owner of the put chooses to exercise it, buy the stock for $K$ and sell it for $S(t)$. Net transaction $S(t) - K + P^a e^{rt} > S(t) + K(e^{rt} - 1) > 0$.
  - If the put expires unused, the profit is $P^a e^{rT} > 0$. 
Proof

- Suppose $P^a < K - S$.
  - Buy the put and the stock (cost $P^a + S$).
  - Immediately exercise the put and sell the stock for $K$.
  - Net transaction $K - P^a - S > 0$ (arbitrage).

- Suppose $P^a > K$.
  - Sell the put and invest proceeds at risk-free rate $r$. Amount due at time $t$ is $P^a e^{rt}$.
  - If the owner of the put chooses to exercise it, buy the stock for $K$ and sell it for $S(t)$. Net transaction $S(t) - K + P^a e^{rt} > S(t) + K(e^{rt} - 1) > 0$.
  - If the put expires unused, the profit is $P^a e^{rT} > 0$. 
Remark: in contrast to American calls, American puts on non-dividend-paying stocks will sometimes be exercised early.

Example
Consider a 12-month American put on a non-dividend paying stock currently worth $15. If the risk-free interest rate is 3.25% per year and the strike price of the put is $470, should the option be exercised early?
Solution

- We were not told the price of the put, but we know \( P^a < K = 470 \).
- If we exercise the put immediately, we gain \( 470 - 15 = 455 \) and invest at the risk-free rate.
- In one year the amount due is \( 455e^{0.0325} = 470.03 > K > P^a \).

Thus the option should be exercised early.
American Calls

Theorem
For a non-dividend-paying stock whose current price is $S$ and for which the American call with a strike price of $K$ and expiry $T$ has a value of $C^a$, satisfies the inequality

$$(S - Ke^{-rT})^+ \leq C^a < S.$$
Theorem
Suppose $T_1 < T_2$ and
- let $C^a(T_i)$ be the value of an American call with expiry $T_i$, and
- let $P^a(T_i)$ be the value of an American put with expiry $T_i$,
then

$$C^a(T_1) \leq C^a(T_2)$$
$$P^a(T_1) \leq P^a(T_2).$$
Proof

Suppose $C^a(T_1) > C^a(T_2)$.

- Buy the option $C^a(T_2)$ and sell the option $C^a(T_1)$. Initial transaction,
  \[ C^a(T_1) > C^a(T_2) > 0 \]

- If the owner of $C^a(T_1)$ chooses to exercise the option, we can exercise the option $C^a(T_2)$. Transaction cost,
  \[ (S(t) - K) - (S(t) - K) = 0. \]

Since we keep the initial transaction profit, arbitrage is present.
Theorem

Suppose $K_1 < K_2$ and

- let $C^a(K_i)$ be the value of an American call with strike price $K_i$, and
- let $P^a(K_i)$ be the value of an American put with strike price $K_i$,

then

\[
C^a(K_2) \leq C^a(K_1) \\
P^a(K_1) \leq P^a(K_2) \\
C^a(K_1) - C^a(K_2) \leq K_2 - K_1 \\
P^a(K_2) - P^a(K_1) \leq K_2 - K_1.
\]
Theorem

Suppose $S_1 < S_2$ and

- let $C^a(S_i)$ be the value of an American call written on a stock whose value is $S_i$, and
- let $P^a(S_i)$ be the value of an American put written on a stock whose value is $S_i$,

then

$$C^a(S_1) \leq C^a(S_2)$$
$$P^a(S_2) \leq P^a(S_1)$$
$$C^a(S_2) - C^a(S_1) \leq S_2 - S_1$$
$$P^a(S_1) - P^a(S_2) \leq S_2 - S_1.$$
Suppose $C^a(S_1) > C^a(S_2)$. 
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Define $x_1 = S_1 - S(0)$ and $x_2 = S_2 - S(0)$. 

Proof (1 of 3)
Suppose $C^a(S_1) > C^a(S_2)$. Ordinarily we would argue to purchase the call $C^a(S_2)$ and sell the call $C^a(S_1)$; however, the stock has only one price for all buyers and sellers, initially $S(0)$.

Define $x_1 = \frac{S_1}{S(0)}$ and $x_2 = \frac{S_2}{S(0)}$. 
Proof (2 of 3)

- Sell $x_1$ options $C^a(S(0))$ where
  \[ x_1 C^a(S(0)) = C^a(S_1) \]
  and buy $x_2$ options $C^a(S(0))$ where
  \[ x_2 C^a(S(0)) = C^a(S_2). \]

Initial transaction is

\[ C^a(S_1) - C^a(S_2) > 0. \]

- If the owner of option $C^a(S_1)$ chooses to exercise it, option $C^a(S_2)$ is exercised as well. Transaction profit is

\[ x_2 (S(t) - K) - x_1 (S(t) - K) = (x_2 - x_1)(S(t) - K) > 0. \]

Arbitrage is present.
Proof (3 of 3)

Suppose $P^a(S_1) - P^a(S_2) > S_2 - S_1$, this is equivalent to the inequality

$$P^a(S_1) + S_1 > P^a(S_2) + S_2.$$ 

- Buy $x_2$ put options $P^a(S(0))$, sell $x_1$ put options $P^a(S(0))$, and buy $x_2 - x_1$ shares of stock. Initial transaction,

$$x_1 P^a(S(0)) - x_2 P^a(S(0)) - (x_2 - x_1)S(0)$$

$$= P^a(S_1) - P^a(S_2) - (S_2 - S_1) > 0.$$ 

- If the owner of put $P^a(S_1)$ chooses to exercise the option, we exercise put $P^a(S_2)$ and sell our $x_2 - x_1$ shares of stock.

$$(x_2 - x_1)S(t) + x_2(K - S(t)) - x_1(K - S(t)) = (x_2 - x_1)K > 0$$

Arbitrage is present.
Binomial Pricing of American Puts

Assumptions:

- Strike price of the American put is $K$,
- Expiry date of the American put is $T > 0$,
- Price of the security at time $t$ with $0 \leq t \leq T$ is $S(t)$,
- Continuously compounded risk-free interest rate is $r$, and
- Price of the security follows a geometric Brownian motion with variance $\sigma^2$. 
Definitions

\( u \): factor by which the stock price may increase during a time step.

\[ u = e^{\sigma \sqrt{\Delta t}} > 1 \]

\( d \): factor by which the stock price may decrease during a time step.

\[ 0 < d = e^{-\sigma \sqrt{\Delta t}} < 1 \]

\( p \): probability of an increase in stock price during a time step.

\[ 0 < p = \frac{1}{2} \left( 1 + \left( \frac{r}{\sigma} - \frac{\sigma}{2} \right) \sqrt{\Delta t} \right) < 1 \]
Illustration

\[ S(T) = u \ S(0) \]

\[ S(T) = d \ S(0) \]

\[ \begin{align*}
S(0) & \quad p \\
& \quad 1 - p \\
S(T) = u S(0) & \\
S(T) = d S(0) &
\end{align*} \]
**Intrinsic Value**

**Observation:** an American put is always worth at least as much as the payoff generated by immediate exercise.

**Definition**
The **intrinsic value** at time $t$ of an American put is the quantity $(K - S(t))^+$. 
Observation: an American put is always worth at least as much as the payoff generated by immediate exercise.

Definition
The intrinsic value at time $t$ of an American put is the quantity $(K - S(t))^+$. The value of an American put is the greater of its intrinsic value and the present value of its expected intrinsic value at the next time step.
One-Step Illustration (1 of 2)
At expiry the American put is worth

\[ P^a(T) = \begin{cases} 
(K - uS(0))^+ & \text{with probability } p, \\
(K - dS(0))^+ & \text{with probability } 1 - p.
\end{cases} \]
At expiry the American put is worth

\[ P^a(T) = \begin{cases} (K - uS(0))^+ & \text{with probability } p, \\ (K - dS(0))^+ & \text{with probability } 1 - p. \end{cases} \]

At \( t = 0 \) the American put is worth

\[ P^a(0) = \max \left\{ (K - S(0))^+, \quad e^{-rT} \left[ p(K - uS(0))^+ + (1 - p)(K - dS(0))^+ \right] \right\} \]

\[ = \max \left\{ (K - S(0))^+, e^{-rT} \mathbb{E} \left[ (K - S(T))^+ \right] \right\} \]

\[ = \max \left\{ (K - S(0))^+, e^{-rT} \mathbb{E} [P^a(T)] \right\}. \]
Two-Step Illustration (1 of 2)
At $t = T/2$, if the put has not been exercised already, an investor will exercise it, if the option is worth more than the present value of the expected value at $t = T$.

$$P^a(T/2) = \max \left\{ (K - S(T/2))^+, e^{-rT/2}E[P^a(T)] \right\}$$
At $t = T/2$, if the put has not been exercised already, an investor will exercise it, if the option is worth more than the present value of the expected value at $t = T$.

$$P^a(T/2) = \max \left\{ (K - S(T/2))^+, e^{-rT/2}E[P^a(T)] \right\}$$

Using the same logic, the value of the put at $t = 0$ is the larger of the intrinsic value at $t = 0$ and the present value of the expected value at $t = T/2$.

$$P^a(0) = \max \left\{ (K - S(0))^+, e^{-rT/2}E[P^a(T/2)] \right\}$$
Example

Suppose the current price of a security is $32, the risk-free interest rate is 10% compounded continuously, and the volatility of Brownian motion for the security is 20%. Find the price of a two-month American put with a strike price of $34 on the security.
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Suppose the current price of a security is $32, the risk-free interest rate is 10% compounded continuously, and the volatility of Brownian motion for the security is 20%. Find the price of a two-month American put with a strike price of $34 on the security.

We will set $\Delta t = 1/12$, then

\[
\begin{align*}
    u & \approx 1.0594 \\
    d & \approx 0.9439 \\
    p & \approx 0.5574.
\end{align*}
\]
Stock Price Lattice

[Diagram with nodes labeled: 35.9168, 33.9019, 30.2048, 28.5103, 32.0000]
Intrinsic Value Lattice

0.0981044

2.5.48969

3.7952

5.48969
Pricing the Put at $t = 1/12$

If $S(1/12) = 33.9019$ then

$$P^a(1/12) = \max \left\{ (34 - 33.9019)^+, \right.$$ 
$$e^{-0.10/12} \left( 0.5574(34 - 35.9168)^+ 
+ (1 - 0.5574)(34 - 32)^+ \right) \right\}$$

$$= 0.8779.$$ 

If $S(1/12) = 30.2048$ then

$$P^a(1/12) = \max \left\{ (34 - 30.2048)^+, \right.$$ 
$$e^{-0.10/12} \left( 0.5574(34 - 32)^+ 
+ (1 - 0.5574)(34 - 28.5103)^+ \right) \right\}$$

$$= 3.7942.$$
Pricing the Put at $t = 0$

\[
P^a(0) = \max \left\{ (34 - 32)^+, \right. \\
\left. e^{-0.10/12} \left[ (0.5574)(0.8779) + (0.4426)(3.7952) \right] \right\} \\
= 2.1513.
\]
American Put Lattice
Summary: \[
\begin{bmatrix}
P^a(t)
(K - S(t))^+
S(t)
\end{bmatrix}
\]
Using a recursive procedure, the value of an American option, for example a put, is given by

\[ P^a(T) = (K - S(T))^+ \]

\[ P^a((n-1)\Delta t) = \max \left\{ (K - S((n-1)\Delta t))^+, e^{-r\Delta t}E[P^a(T)] \right\} \]

\[ P^a((n-2)\Delta t) = \max \left\{ (K - S((n-2)\Delta t))^+, e^{-r\Delta t}E[P^a((n-1)\Delta t)] \right\} \]

\[ \vdots \]

\[ P^a(0) = \max \left\{ (K - S(0))^+, e^{-r\Delta t}E[P^a(\Delta T)] \right\} . \]
Early Exercise for American Calls

If a stock pays a dividend during the life of an American call option, it may be advantageous to exercise the call early so as to collect the dividend.

Example
Suppose a stock is currently worth $150 and has a volatility of 25% per year. The stock will pay a dividend of $15 in two months. The risk-free interest rate is 3.25%. Find the prices of two-month European and American call options on the stock with strikes prices of $150.
If $\Delta t = 1/12$, then

\[
\begin{align*}
    u &= e^{\sigma \sqrt{\Delta t}} \approx 1.07484 \\
    d &= e^{-\sigma \sqrt{\Delta t}} \approx 0.930374 \\
    p &= \frac{1}{2} \left( 1 + \left( \frac{r}{\sigma} - \frac{\sigma}{2} \right) \sqrt{\Delta t} \right) \approx 0.500722.
\end{align*}
\]
Solution (2 of 3) Stock Prices

150 161.226 139.556 158.291 135. 114.839
Solution (3 of 3) Call Prices