Annuities and Loans

MATH 472 *Financial Mathematics*

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Objective

In this lesson we will:

- review the finite geometric series,
- develop relationships between the principle amount of a loan, the interest rate, the level payment, and the number of payments,
- develop formulas for creating an annuity,
- see how inflation affects the effective annual interest rate,
- learn how to amortize a loan.

Geometric Series

Theorem

If $a \neq 1$ *then* $S = 1 + a + a^2 + \cdots + a^n = \dfrac{1 - a^{n+1}}{1 - a}$.

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Proof.

Let $S = 1 + a + a^2 + \cdots + a^n$ then $aS = a + a^2 + \cdots + a^n + a^{n+1}$ and

$$\begin{aligned} S - aS &= (1 + a + \cdots + a^n) - (a + a^2 + \cdots + a^{n+1}) \\ S(1 - a) &= 1 - a^{n+1} \\ S &= \frac{1 - a^{n+1}}{1 - a} \end{aligned}$$

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Corollary

If $a \neq 1$ *then* $\sum_{k=1}^{n} a^k = \dfrac{a - a^{n+1}}{1 - a}$.

Loan Payments (1 of 2)

Suppose a loan of amount P will be paid back discretely (n times per year) over t years. All payments will be the same amount. The unpaid portion of the loan is charged interest at annual rate r compounded n times per year. What is the constant, level payment x?

Hint: the present value of all the payments should equal the amount borrowed.

Loan Payments (2 of 2)

If the first payment must be made at the end of the first compounding period, then the present value of all the payments is

$$
\begin{aligned}
P &= x(1+\frac{r}{n})^{-1} + x(1+\frac{r}{n})^{-2} + \cdots + x(1+\frac{r}{n})^{-nt} \\
&= x\sum_{k=1}^{nt}\left(1+\frac{r}{n}\right)^{-k} = x\sum_{k=1}^{nt}\left(\frac{n}{n+r}\right)^{k} \\
&= x\frac{\frac{n}{n+r} - \left(\frac{n}{n+r}\right)^{nt+1}}{1-\frac{n}{n+r}} \\
P &= \frac{x\,n}{r}\left(1-\left(\frac{n}{n+r}\right)^{n\,t}\right).
\end{aligned}
$$

Example

If a person borrows \$25,000 for five years at an interest rate of 4.99% compounded monthly and makes equal monthly payments, what is the monthly payment?

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Solution

$$\begin{aligned} x &= \frac{P\,r}{n}\left(1-\left(\frac{n}{n+r}\right)^{n\,t}\right)^{-1} \\ &= \frac{(25000)(0.0499)}{12}\left(1-\left(\frac{12}{12+0.0499}\right)^{(12)(5)}\right)^{-1} \\ &\approx \$471.67 \end{aligned}$$

Retirement Savings

Suppose a person is 25 years of age now and plans to retire at age 65. For the next 40 years they plan to invest a portion of their monthly income in securities which earn interest at the annual rate of 10% compounded monthly. After retirement the person plans on receiving a monthly payment (an annuity) in the absolute amount of $1500 for 30 years. How much should be set aside monthly for retirement?

Solution (1 of 2)

The present value of all funds invested for retirement should equal the present value of all funds taken out during retirement.

- If x is invested at the end of each month for the next 40 years, then the present value of all the investments is

$$P = x \sum_{k=1}^{480} \left(1 + \frac{0.10}{12}\right)^{-k} = x \sum_{k=1}^{480} \left(\frac{12}{12.10}\right)^{k} = 117.765x.$$

- If \$1500 is received from the annuity each month for 30 years beginning 481 months from now, the present value of all the payments is

$$\begin{aligned} P &= 1500 \left(1 + \frac{0.10}{12}\right)^{-480} \sum_{k=1}^{360} \left(1 + \frac{0.10}{12}\right)^{-k} \\ &= 1500 \left(\frac{12}{12.10}\right)^{480} \sum_{k=1}^{360} \left(\frac{12}{12.10}\right)^{k} = 3182.94 \end{aligned}$$

Solution (2 of 2)

Equating the two present values produces an investment amount of

$$117.765x = 3182.94 \iff x \approx \$27.03.$$

Adjusting for Inflation

Definition

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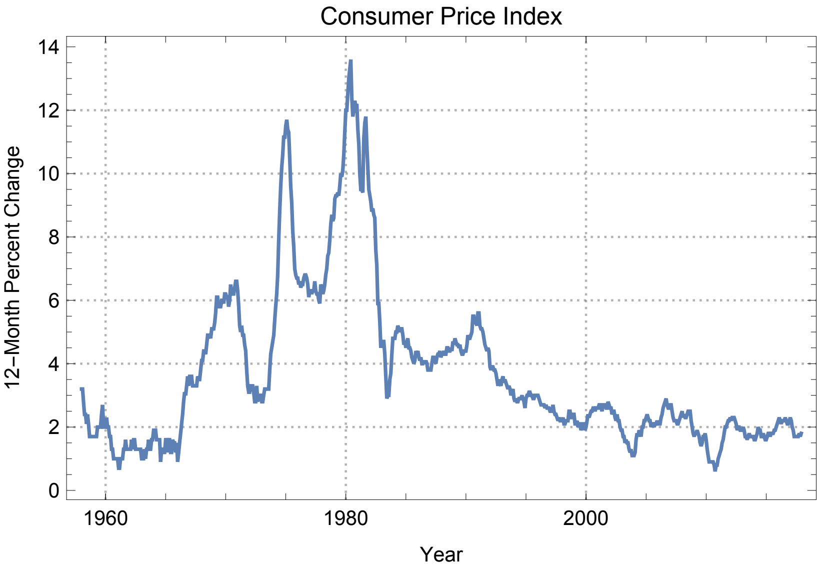
How does inflation (measured at an annual rate i) affect the value of deposits earning interest?

Illustration



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- The difference $(1 + r) - (1 + i) = r - i$ will be the real rate of growth in the unit of currency invested now.
- This return on saving will not be earned until one year from now. The present value of $r - i$ under inflation rate i is

$$r_i = \frac{r - i}{1 + i}.$$

Retirement Annuity (revisited)

Suppose a person is 25 years of age now and plans to retire at age 65. For the next 40 years they plan to invest a portion of their monthly income in securities which earn interest at the rate of 10% compounded monthly. After retirement the person plans on receiving a monthly payment (an annuity) in the absolute amount of $1500 for 30 years. How much should be set aside monthly for retirement if the annual inflation rate is 3%?

Solution

The inflation adjusted return on savings is

$$r_i = \frac{r - i}{1 + i} = \frac{0.10 - 0.03}{1 + 0.03} \approx 0.0679612.$$

Using this value in place of r in the previous example we have

$$\begin{aligned} P &= x \sum_{k=1}^{480} \left(\frac{12}{12.0679612} \right)^k = 164.832x \\ P &= 1500 \left(\frac{12}{12.0679612} \right)^{480} \sum_{k=1}^{360} \left(\frac{12}{12.0679612} \right)^k = 15,303.4 \\ x &\approx \$92.84. \end{aligned}$$

Mortgage Amortization (1 of 4)

Suppose a person takes out a mortgage loan in the amount of L and will make n equal monthly payments of amount x where the annual interest rate is r compounded monthly.

1. Express x as a function of L, r, and n.
2. After the jth month, how much of the original amount borrowed remains?
3. How much of the jth payment goes to interest and how much goes to pay down the amount borrowed?

Mortgage Amortization (2 of 4)

The sum of the present values of all the payments must equal the amount loaned.

$$\begin{aligned} L &= \sum_{k=1}^{n} \frac{x}{(1+r/12)^k} \\ &= x \sum_{k=1}^{n} \left(\frac{12}{12+r} \right)^k \\ &= x \frac{\frac{12}{12+r} - \left(\frac{12}{12+r} \right)^{n+1}}{1 - \frac{12}{12+r}} \\ &= \frac{12x}{r} \left(1 - \left(\frac{12}{12+r} \right)^n \right) \end{aligned}$$

Mortgage Amortization (3 of 4)

The outstanding balance on the loan immediately after the jth monthly payment will be the sum of the present values of the remaining $n - j$ payments. Let L_j denote the outstanding balance immediately after the jth payment, then

$$
\begin{aligned}
L_j &= \sum_{k=1}^{n-j} \frac{x}{\left(1 + \frac{r}{12}\right)^k} \\
&= \frac{12x}{r}\left(1 - \left(\frac{12}{12+r}\right)^{n-j}\right).
\end{aligned}
$$

Mortgage Amortization (4 of 4)

- The jth payment of x can be divided into an **interest payment** I_j and a **principal repayment** P_j.
- I_j is the interest earned by the principal remaining after the $(j-1)$st loan payment.

$$I_j = \left(\frac{r}{12}\right) L_{j-1} = x\left(1 - \left(\frac{12}{12+r}\right)^{n-j+1}\right).$$

- P_j is the portion of x not consumed by the interest payment.

$$P_j = x - I_j = x\left(\frac{12}{12+r}\right)^{n-j+1}.$$

Mortgage Example

Suppose \$284,000 is borrowed to purchase a house. The annual interest rate of the mortgage is 4.75% compounded monthly and the term of the mortgage is 15 years.

1. What is the regular monthly payment?
2. What is the balance on the outstanding principal after the 99th payment?
3. How much of the 100th payment goes to pay interest?
4. How much of the 100th payment goes to repay principal?

Solution (1 of 2)

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1. What is the regular monthly payment?

$$x = \frac{L\,r}{12}\left(1 - \left(\frac{12}{12+r}\right)^{n}\right)^{-1}$$
$$= \frac{(284000)(0.0475)}{12}\left(1 - \left(\frac{12}{12.0475}\right)^{180}\right)^{-1} = \$2,209.04$$

2. What is the balance on the outstanding principal after the 99th payment?

Solution (1 of 2)

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2. What is the balance on the outstanding principal after the 99th payment?

$$L_{99} = \frac{12(2209.04)}{0.0475}\left(1-\left(\frac{12}{12.0475}\right)^{180-99}\right) = \$152,825.70$$

Solution (2 of 2)

3. How much of the 100th payment goes to pay interest?

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$$I_{100} = \left(\frac{0.0475}{12}\right)(152825.70) = \$604.94$$

4. How much of the 100th payment goes to repay principal?

Solution (2 of 2)

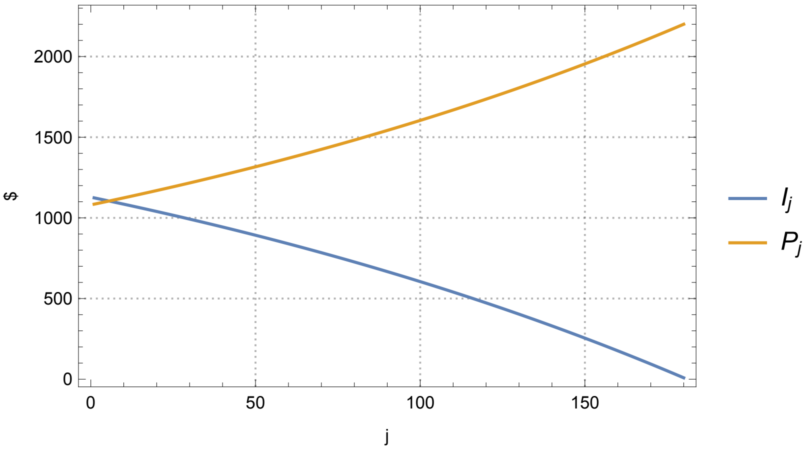
3. How much of the 100th payment goes to pay interest?

$$I_{100} = \left(\frac{0.0475}{12}\right)(152825.70) = \$604.94$$

4. How much of the 100th payment goes to repay principal?

$$P_{100} = 2209.04 - 604.94 = \$1,604.10$$

Illustration



Homework

- Read Section 1.4.
- Exercises: 8–10.

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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