

Arbitrage

MATH 472 *Financial Mathematics*

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2018

Efficient Market Hypothesis

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- ▶ prices will adjust to any new information nearly instantaneously,
- ▶ an investor cannot outperform the market using known information except through luck.

Arbitrage

- ▶ **Arbitrage** arises from mis-priced financial instruments or commodities.
- ▶ To take advantage of mis-priced items, an investor will have to purchase and sell identical items (or interchangeable items) in a short time (nearly simultaneously).
- ▶ The Efficient Market Hypothesis implies that arbitrage situations are usually short-lived (why?).

Two Types of Arbitrage

- Type A:** a trading strategy which results in an initial positive cash flow to the investor with no risk of future loss.
- Type B:** a trading strategy requiring no initial cash investment, has no risk of future loss, and has a positive probability of profit.

Simple Arbitrage Situation

Devise a trading strategy for the following situation which results in a positive profit to the trader.

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- ▶ CostCo sells 100 stamps for \$48.75.
- ▶ USPS sell 100 stamps for \$49.00.

Purchase the stamps from CostCo and sell them outside the local post office. Each trade generates \$0.25 (which might not seem like much, but do it a million times).

Elimination of Arbitrage

- ▶ Suppose we can invest \$1000 for two years at an annually compounded interest rate of 3.25%.
- ▶ Suppose we can invest \$1000 for one year at an annually compounded interest rate of 2.75% and then lend the amount due at the annually compounded rate $r\%$ for a second year.

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- ▶ If r is not as determined above, what arbitrage opportunities are available? If $(1 + 0.0325)^2 < (1 + 0.0275)(1 + r)$ then borrow at 3.25% for two years and lend at $(1 + 0.0275)(1 + r)$ for two years.

Intuitive Idea

Imagine we will bet on the outcome of an experiment.

The **Arbitrage Theorem** states that either the probabilities of the outcomes are such that

- ▶ all bets are fair, or
- ▶ there is a betting scheme which produces a positive gain independent of the outcome of the experiment.

Odds

The **odds against** an outcome X are related to probabilities of the outcome according to the formula:

$$n : m \text{ against} \iff \mathbb{P}(X) = \frac{m}{m+n}.$$

The **odds for** an outcome X are related to probabilities of the outcome according to the formula:

$$n : m \text{ in favor} \iff \mathbb{P}(X) = \frac{n}{m+n}.$$

Wagering

For a wager of m dollars on a event X with odds against of $n : m$,

- ▶ if X occurs, we win $n + m$ dollars (our initial investment of m dollars plus n dollars in profit),
- ▶ if X does not occur, we lose our investment of m dollars.

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Ignoring several complicating factors, suppose the following amounts were wagered on each of six horses to win a race.

Horse	1	2	3	4	5	6
Amount	\$70	\$22	\$20	\$98	\$50	\$68

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Questions:

- ▶ What are the odds against each horse?
- ▶ How much profit would a unit bet on winning horse 6 generate?

Solution

- ▶ The total amount wagered on all horses is $W = \$328$.
- ▶ The odds on horse i are calculated as the quotient of W divided by the amount wagered on horse i .

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Horse	1	2	3	4	5	6
Amount	\$70	\$22	\$20	\$98	\$50	\$68
Odds	3.69 : 1	13.91 : 1	15.40 : 1	2.35 : 1	5.56 : 1	3.82 : 1
$\mathbb{P}(\text{win})$	0.2134	0.0671	0.0610	0.2988	0.1524	0.2073

Example (1 of 2)

Suppose the odds against player A defeating player B in a tennis match are 3 : 1 and the odds against player B defeating player A are 1 : 1.

$$\mathbb{P}(A \text{ wins}) = 0.25 \quad \text{and} \quad \mathbb{P}(B \text{ wins}) = 0.5$$

Determine a betting strategy which guarantees a positive net profit regardless of the outcome of the tennis match.

Example (2 of 2)

Betting strategy: wager \$1 on player A and \$2 on player B.

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- ▶ If A wins: gain \$3 on the first bet and lose \$2 on the second, net gain of \$1.
- ▶ If B wins: lose \$1 on the first bet and gain \$2 on the second, net gain of \$1.

There is a positive payoff no matter which player wins.

Real-life Arbitrage (1 of 2)

2006 Winter Olympics, Turin, Italy

- ▶ Online casino, SportingUSA.com (now out of business) offered 2.5 : 1 odds against Denmark winning medals.
- ▶ Online casino Bet365.com (still in business) offered 1.875 : 1 odds Denmark would win at least one medal.

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- ▶ Suppose a bettor had \$1,000 to bet and wagered \$500 at each casino.
- ▶ Is it possible to guarantee a positive profit?

Real-life Arbitrage (2 of 2)

- ▶ If Denmark does not win a medal the bettor receives

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- ▶ In the worst case, the bettor has invested \$1,000 and received \$1,437.50.
- ▶ Denmark did not medal in 2006.

Introduction to Linear Programming

Linear programming is a branch of mathematics concerned with optimizing a linear function of several variables subject to some set of constraints (linear equalities or inequalities) on the variables.

The proof of the **Arbitrage Theorem** requires some familiarity with linear programming.

Example (1 of 3)

A bank may invest its deposits in loans which earn 6% interest per year and in the purchase of stocks which increase in value by 13% per year. Any un-invested amount is simply held by the bank. Suppose that government regulations require that the bank invest no more than 60% of its deposits in stocks and must keep 10% of its deposits on hand in the form of cash. As a good business practice the bank wishes to devote at least 25% of its deposits to loans. Determine how the bank should allocate its capital so as to maximize the total return on its investments.

Example (2 of 3)

- ▶ Assume the bank can invest a fraction x in loans and fraction y in stocks.

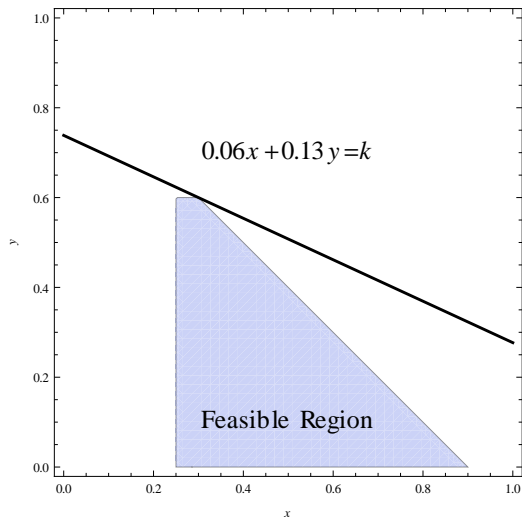
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- ▶ The total return is therefore $0.06x + 0.13y$.
- ▶ The constraints are:
 - ▶ $0.25 \leq x$
 - ▶ $0 \leq y \leq 0.60$
 - ▶ $x + y \leq 0.90$

Example (3 of 3)



Optimal return of $k = 0.096$ occurs when $x = 0.3$ and $y = 0.6$.

Decision Variables and Objective Functions

If \mathbf{c} and \mathbf{x} are vectors with n components each, the notation

$$\mathbf{c}^T \mathbf{x} = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

represents a weighted sum of the components of \mathbf{x} with the weights being the components of \mathbf{c} .

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Remarks:

- ▶ The components of \mathbf{x} are sometimes called **decision variables**.
- ▶ The weighted sum $\mathbf{c}^T \mathbf{x}$ is called an **objective function**.

Constraints

Constraints on the decision variables will be expressed in the form $\mathbf{a}^T \mathbf{x} \leq z$ where \mathbf{a} is a vector of n components and z is a scalar.

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All relationships can be expressed using \leq .

$$\mathbf{a}^T \mathbf{x} \geq z \iff (-\mathbf{a})^T \mathbf{x} \leq -z$$

$$\mathbf{a}^T \mathbf{x} = z \iff \mathbf{a}^T \mathbf{x} \leq z \quad \text{and} \quad (-\mathbf{a})^T \mathbf{x} \leq -z$$

Vector Comparisons

We write $\mathbf{u} < \mathbf{v}$ if $u_i < v_i$ for $i = 1, 2, \dots, n$.

Similarly for

- ▶ $\mathbf{u} > \mathbf{v}$,
- ▶ $\mathbf{u} \leq \mathbf{v}$, and
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If $\mathbf{0}$ denotes the **zero vector** then $\mathbf{x} \geq \mathbf{0}$ is an example of a **sign constraint**.

Optimization

When solving a linear program, we will

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Remark: the processes of maximizing and minimizing $\mathbf{c}^T \mathbf{x}$ are equivalent in the sense that $\mathbf{c}^T \mathbf{x}$ is a maximum if and only if $(-\mathbf{c})^T \mathbf{x}$ is a minimum.

Systems of Constraints

Suppose there are m inequality constraints:

$$\begin{aligned} \mathbf{a}_1^T \mathbf{x} &\leq b_1 \\ \mathbf{a}_2^T \mathbf{x} &\leq b_2 \\ &\vdots \\ \mathbf{a}_m^T \mathbf{x} &\leq b_m \end{aligned}$$

we may express this in matrix form as

$$\mathbf{A} \mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \mathbf{b}.$$

Forms of Linear Programs

There are at least three equivalent forms of linear programs:

Standard form: decision variables $\mathbf{x} \geq \mathbf{0}$, constraints $A\mathbf{x} = \mathbf{b}$.

Canonical form: decision variables $\mathbf{x} \geq \mathbf{0}$, constraints
 $A\mathbf{x} = \mathbf{b} \geq \mathbf{0}$.

Symmetric form: decision variables $\mathbf{x} \geq \mathbf{0}$, constraints $A\mathbf{x} \leq \mathbf{b}$.

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Symmetric form: decision variables $\mathbf{x} \geq \mathbf{0}$, constraints $A\mathbf{x} \leq \mathbf{b}$.

Remark: since any one of the three forms can be re-cast as any of the other forms, we are free to work with the most convenient formulation in any given context.

Equivalence of Symmetric and Standard Forms

Given the symmetric linear program: maximize $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$, introduce **slack variables**.

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Given the symmetric linear program: maximize $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$, introduce **slack variables**.

1. If $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ augment with m slack variables $\hat{x}_{n+j} = b_j - \sum_{i=1}^n a_{ji}x_i$ for $j = 1, 2, \dots, m$ to form decision variable:

$$\bar{\mathbf{x}} = \langle \mathbf{x}, \hat{\mathbf{x}} \rangle = \langle x_1, x_2, \dots, x_n, \hat{x}_{n+1}, \hat{x}_{n+2}, \dots, \hat{x}_{n+m} \rangle.$$

2. If $A \in \mathbb{R}^{m \times n}$ then augment the columns of A with the $m \times m$ identity matrix.

$$\left[A \mid I_m \right] \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} = \bar{A}\bar{\mathbf{x}} = \mathbf{b}$$

3. Augment \mathbf{c} with m zeros, then $\bar{\mathbf{c}}^T \bar{\mathbf{x}} = \mathbf{c}^T \mathbf{x}$.

Equivalence of Standard and Symmetric Forms

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Given the standard linear program: maximize $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$, introduce inequality constraints.

1. $A\mathbf{x} = \mathbf{b}$ if and only if $A\mathbf{x} \leq \mathbf{b}$ and $-A\mathbf{x} \leq -\mathbf{b}$.
2. Augment the rows of matrix A with the rows of matrix $-A$.

$$\begin{bmatrix} A \\ -A \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$$

3. The weighted sum remains the same.

General Linear Program

The most flexible statement of a linear program relaxes the non-negativity of the decision variables and mixes the equations and inequalities of the constraints.

A linear program of the form: maximize $\mathbf{c}^T \mathbf{x}$ subject to

$$A\mathbf{x} \leq \mathbf{b}$$

$$\hat{A}\mathbf{x} \geq \hat{\mathbf{b}}$$

$$\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}$$

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is called a **general linear program**.

Remark: every standard, canonical, or symmetric linear program is trivially a general linear program. The converse is also true.

Positive and Negative Parts

Definition

The **positive part** of real number x is denoted x^+ and is

$$x^+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The **negative part** of real number x is denoted x^- and is

$$x^- = \begin{cases} -x & \text{if } x \leq 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Remark: this definition can be applied component-wise to real vectors.

Equivalence of General and Symmetric Programs

Given a linear program in general form, we can construct a symmetric linear program.

1. If the decision vector $\mathbf{x} \in \mathbb{R}^n$ is unrestricted in sign, create a new decision vector $\langle \mathbf{x}^+, \mathbf{x}^- \rangle \in \mathbb{R}^{2n}$.

$$\langle \mathbf{x}^+, \mathbf{x}^- \rangle \geq \mathbf{0}$$

2. Create a new vector of weights $\langle \mathbf{c}, -\mathbf{c} \rangle$.

$$\langle \mathbf{c}, -\mathbf{c} \rangle^T \langle \mathbf{x}^+, \mathbf{x}^- \rangle = \mathbf{c}^T (\mathbf{x}^+ - \mathbf{x}^-) = \mathbf{c}^T \mathbf{x}$$

Equivalence of General and Symmetric Programs

3. The system of constraints is re-written in inequality form:

$$\begin{bmatrix} A & -A \\ -\hat{A} & \hat{A} \\ \tilde{A} & -\tilde{A} \\ -\tilde{A} & \tilde{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\hat{\mathbf{b}} \\ \tilde{\mathbf{b}} \\ -\tilde{\mathbf{b}} \end{bmatrix} .$$

Feasible Vectors and Cost Functions

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Vector \mathbf{x} is **feasible** if $\mathbf{x} \geq \mathbf{0}$ and $A\mathbf{x} \leq \mathbf{b}$.

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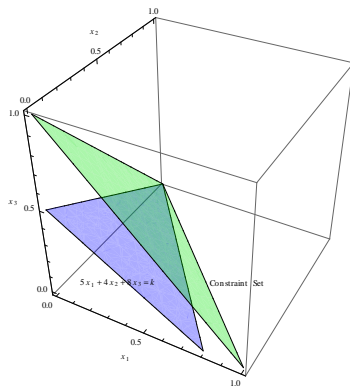
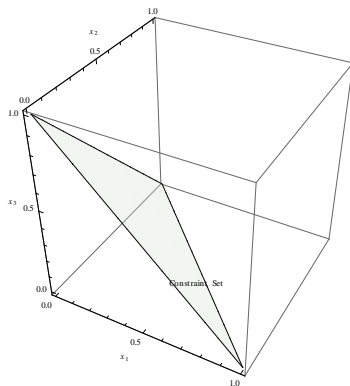
Vector \mathbf{x} is an **optimal solution** if \mathbf{x} is feasible and maximizes the cost function.

Example (1 of 2)

Use the notion of the intersection of planes in \mathbb{R}^3 to minimize $5x_1 + 4x_2 + 8x_3$ subject to $x_1 + x_2 + x_3 = 1$ and \mathbf{x} is feasible.

Remark: this is a linear program stated in standard form.

Example (2 of 2)



The cost function has a minimum of 4 at $\mathbf{x} = \langle 0, 1, 0 \rangle$.

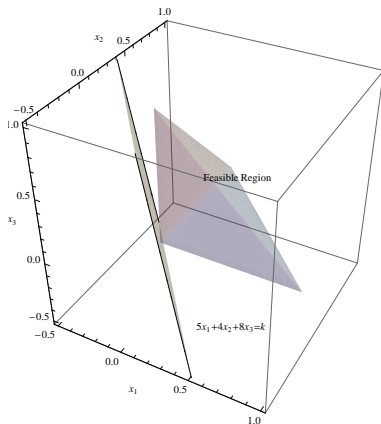
Example (1 of 2)

Use the notion of the intersection of planes in \mathbb{R}^3 to minimize $5x_1 + 4x_2 + 8x_3$ subject to $x_1 + x_2 + x_3 \leq 1$ and \mathbf{x} is feasible.

Remark: this linear program is stated in symmetric form.

Example (2 of 2)

If the constraints are $x_1 + x_2 + x_3 \leq 1$ and \mathbf{x} feasible, then the set of points where the solution must be found would resemble a tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.



The cost function has a minimum of 0 at $\mathbf{x} = \langle 0, 0, 0 \rangle$.

Dual Problems

For every linear programming problem of the type discussed earlier, there is an associated problem known as its **dual**. Henceforth the original problem will be known as the **primal**. These paired optimization problems are related in the following ways.

Primal: Maximize $\mathbf{c}^T \mathbf{x}$ subject to $A \mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Dual: Minimize $\mathbf{b}^T \mathbf{y}$ subject to $A^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

Observations

Primal: Maximize $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Dual: Minimize $\mathbf{b}^T \mathbf{y}$ subject to $A^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

Note:

1. the process of maximization in the primal is replaced with the process of minimization in the dual,
2. the unknown of the dual is a vector \mathbf{y} with m components,
3. the vector \mathbf{b} moves from the constraint of the primal to the cost function of the dual,
4. the vector \mathbf{c} moves from the cost of the primal to the constraint of the dual,
5. the constraints of the dual are inequalities and there are n of them.

Equivalences and Duals (1 of 5)

General Linear Program

Maximize $\mathbf{c}^T \mathbf{x}$ subject to

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ \hat{A}\mathbf{x} &\geq \hat{\mathbf{b}} \\ \tilde{A}\mathbf{x} &= \tilde{\mathbf{b}}. \end{aligned}$$

Symmetric Linear Program

Maximize $\langle \mathbf{c}, -\mathbf{c} \rangle^T \langle \mathbf{x}^+, \mathbf{x}^- \rangle$
subject to

$$\begin{bmatrix} A & -A \\ -\hat{A} & \hat{A} \\ \tilde{A} & -\tilde{A} \\ -\tilde{A} & \tilde{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\hat{\mathbf{b}} \\ \tilde{\mathbf{b}} \\ -\tilde{\mathbf{b}} \end{bmatrix}.$$

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$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ \hat{A}\mathbf{x} &\geq \hat{\mathbf{b}} \\ \tilde{A}\mathbf{x} &= \tilde{\mathbf{b}}. \end{aligned}$$

Symmetric Linear Program

Maximize $\langle \mathbf{c}, -\mathbf{c} \rangle^T \langle \mathbf{x}^+, \mathbf{x}^- \rangle$
subject to

$$\begin{bmatrix} A & -A \\ -\hat{A} & \hat{A} \\ \tilde{A} & -\tilde{A} \\ -\tilde{A} & \tilde{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\hat{\mathbf{b}} \\ \tilde{\mathbf{b}} \\ -\tilde{\mathbf{b}} \end{bmatrix}.$$

Now formulate the dual of the Symmetric Linear Program.

Equivalences and Duals (2 of 5)

Dual: minimize $\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$ subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T & -\tilde{A}^T \\ -A^T & \hat{A}^T & -\tilde{A}^T & \tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}}^+ \\ \tilde{\mathbf{y}}^- \end{bmatrix} \geq \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \end{bmatrix},$$

with $\mathbf{y} \geq \mathbf{0}$, $\hat{\mathbf{y}} \geq \mathbf{0}$, $\tilde{\mathbf{y}}^+ \geq \mathbf{0}$, and $\tilde{\mathbf{y}}^- \geq \mathbf{0}$.

Equivalences and Duals (3 of 5)

Let $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}^+ - \tilde{\mathbf{y}}^-$ and then $\tilde{\mathbf{y}}$ is unrestricted in sign and the dual problem can be restated as

minimize $\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$ subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \\ -A^T & \hat{A}^T & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} \geq \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \end{bmatrix}.$$

Equivalences and Duals (3 of 5)

Let $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}^+ - \tilde{\mathbf{y}}^-$ and then $\tilde{\mathbf{y}}$ is unrestricted in sign and the dual problem can be restated as

minimize $\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$ subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \\ -A^T & \hat{A}^T & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} \geq \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \end{bmatrix}.$$

Remark: the constraints are

$$\begin{aligned} A^T \mathbf{y} - \hat{A}^T \hat{\mathbf{y}} + \tilde{A}^T \tilde{\mathbf{y}} &\geq \mathbf{c} \\ -A^T \mathbf{y} + \hat{A}^T \hat{\mathbf{y}} - \tilde{A}^T \tilde{\mathbf{y}} &\geq -\mathbf{c} \end{aligned}$$

Equivalences and Duals (3 of 5)

Let $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}^+ - \tilde{\mathbf{y}}^-$ and then $\tilde{\mathbf{y}}$ is unrestricted in sign and the dual problem can be restated as

minimize $\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$ subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \\ -A^T & \hat{A}^T & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} \geq \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \end{bmatrix}.$$

Remark: the constraints are

$$\begin{aligned} A^T \mathbf{y} - \hat{A}^T \hat{\mathbf{y}} + \tilde{A}^T \tilde{\mathbf{y}} &\geq \mathbf{c} \\ -A^T \mathbf{y} + \hat{A}^T \hat{\mathbf{y}} - \tilde{A}^T \tilde{\mathbf{y}} &\geq -\mathbf{c} \end{aligned}$$

which implies

$$A^T \mathbf{y} - \hat{A}^T \hat{\mathbf{y}} + \tilde{A}^T \tilde{\mathbf{y}} = \mathbf{c}$$

Equivalences and Duals (4 of 5)

Symmetric Linear Program

Maximize $\langle \mathbf{c}, -\mathbf{c} \rangle^T \langle \mathbf{x}^+, \mathbf{x}^- \rangle$
subject to

$$\begin{bmatrix} A & -A \\ -\hat{A} & \hat{A} \\ \tilde{A} & -\tilde{A} \\ -\tilde{A} & \tilde{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\hat{\mathbf{b}} \\ \tilde{\mathbf{b}} \\ -\tilde{\mathbf{b}} \end{bmatrix}.$$

Dual Linear Program

Minimize $\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$
subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} = \mathbf{c}.$$

Equivalences and Duals (5 of 5)

Primal Linear Program

Maximize $\mathbf{c}^T \mathbf{x}$ subject to

$$A\mathbf{x} \leq \mathbf{b}$$

$$\hat{A}\mathbf{x} \geq \hat{\mathbf{b}}$$

$$\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}.$$

Dual Linear Program

Minimize

$$\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$$

subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} = \mathbf{c}.$$

Equivalences and Duals (5 of 5)

Primal Linear Program

Maximize $\mathbf{c}^T \mathbf{x}$ subject to

$$A\mathbf{x} \leq \mathbf{b}$$

$$\hat{A}\mathbf{x} \geq \hat{\mathbf{b}}$$

$$\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}.$$

Dual Linear Program

Minimize

$$\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$$

subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} = \mathbf{c}.$$

Remark: unrestricted decision variables in the primal (dual) problem induce equality constraints in the dual (primal) problem.

Dual of the Dual

Theorem

The dual of the dual is the primal.

Proof

Starting with the dual problem,

$$\text{Minimize } \mathbf{b}^T \mathbf{y} \quad \text{subject to } A^T \mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0}.$$

We can re-write the dual in general form,

$$\text{Maximize } (-\mathbf{b})^T \mathbf{y} \quad \text{subject to } (-A)^T \mathbf{y} \leq -\mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0}.$$

Now the dual of this problem (*i.e.*, the dual of the dual) is

$$\text{Minimize } (-\mathbf{c})^T \mathbf{x} \quad \text{subject to } ((-A)^T)^T \mathbf{x} \geq -\mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}.$$

This problem is logically equivalent to the problem

$$\text{Maximize } \mathbf{c}^T \mathbf{x} \quad \text{subject to } A \mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0},$$

which is the primal problem.

Weak Duality Theorem

Theorem (Weak Duality Theorem)

If \mathbf{x} and \mathbf{y} are the feasible solutions of the primal and dual problems respectively, then $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$. If $\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y}$ then these solutions are optimal for their respective problems.

Proof of Weak Duality Theorem

Feasible solutions to the primal and the dual problems must satisfy the constraints $A\mathbf{x} \leq \mathbf{b}$ with $\mathbf{x} \geq \mathbf{0}$ (for the primal problem) and $A^T\mathbf{y} \geq \mathbf{c}$ with $\mathbf{y} \geq \mathbf{0}$ (for the dual). Multiply the constraint in the dual by \mathbf{x}^T

$$\mathbf{x}^T A^T \mathbf{y} \geq \mathbf{x}^T \mathbf{c} \iff \mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T A \mathbf{x}.$$

Multiply the constraint in the primal by \mathbf{y}^T

$$\mathbf{y}^T A \mathbf{x} \leq \mathbf{y}^T \mathbf{b} = \mathbf{b}^T \mathbf{y}.$$

Directions of the inequalities are preserved because $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$. Combining these last two inequalities produces

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T A \mathbf{x} \leq \mathbf{b}^T \mathbf{y}.$$

Therefore we have $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$. If $\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y}$ then \mathbf{x} and \mathbf{y} must be optimal since no \mathbf{x} can make $\mathbf{c}^T \mathbf{x}$ larger than $\mathbf{b}^T \mathbf{y}$ and no \mathbf{y} can make $\mathbf{b}^T \mathbf{y}$ smaller than $\mathbf{c}^T \mathbf{x}$.

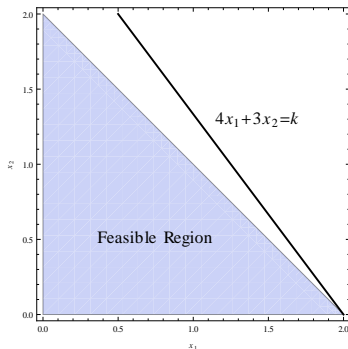
Example (1 of 2)

Primal: Maximize $4x_1 + 3x_2$ subject to $x_1 + x_2 \leq 2$ and $x_1, x_2 \geq 0$.

Dual: Minimize $2y_1$ subject to $y_1 \geq 3$, $y_1 \geq 4$, and $y_1 \geq 0$.

Example (2 of 2)

The minimum value of y_1 subject to the constraints must be $y_1 = 4$. According to the Weak Duality Theorem then the minimum of the cost function of the primal must be at least $2y_1 = 8$. Applying the level set argument as before, the largest value of k for which the level set $4x_1 + 3x_2 = k$ intersects the set of feasible points for the primal is $k = 8$.



Complementary Slackness

Theorem

Optimality in the primal and dual problems requires either $x_j = 0$ or $(A^T \mathbf{y})_j = c_j$ for each $j = 1, \dots, n$.

Proof

When \mathbf{x} and \mathbf{y} are optimal for their respective problems then

$$\mathbf{b}^T \mathbf{y} = \mathbf{y}^T \mathbf{A} \mathbf{x} = \mathbf{c}^T \mathbf{x}$$

$$(\mathbf{y}^T \mathbf{A} - \mathbf{c}^T) \mathbf{x} = 0$$

$$(\mathbf{A}^T \mathbf{y} - \mathbf{c})^T \mathbf{x} = 0.$$

Since $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{A}^T \mathbf{y} - \mathbf{c} \geq \mathbf{0}$ then vector \mathbf{x} must be zero in every component for which vector $\mathbf{A}^T \mathbf{y} - \mathbf{c}$ is positive and vice versa.

Example (1 of 4)

Primal: Maximize $\mathbf{c}^T \mathbf{x} = -3x_1 + 2x_2 - x_3 + 3x_4$ subject to $\mathbf{x} \geq \mathbf{0}$ and

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Example (2 of 4)

Dual: Minimize $\mathbf{b}^T \mathbf{y} = 5y_1 + 3y_2$ subject to

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} -3 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

Example (2 of 4)

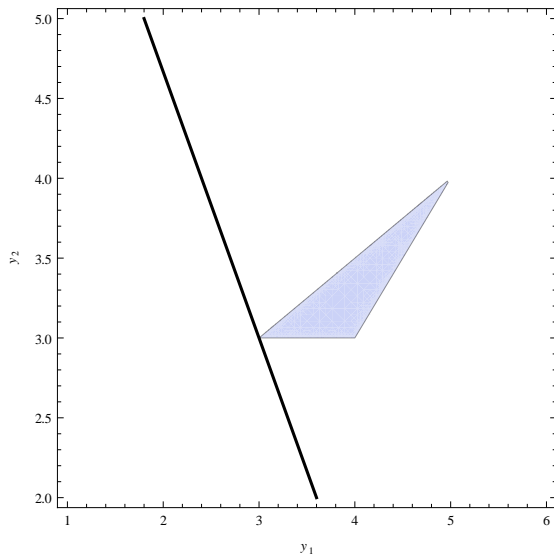
Dual: Minimize $\mathbf{b}^T \mathbf{y} = 5y_1 + 3y_2$ subject to

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} -3 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

Expressed as a system of inequalities, these constraints are

$$\begin{aligned} y_1 - 2y_2 &\geq -3 \\ y_1 &\geq 2 \\ -y_1 + y_2 &\geq -1 \\ y_2 &\geq 3. \end{aligned}$$

Example (3 of 4)



Optimal solution is at $(y_1, y_2) = (3, 3)$ and has value 24.

Example (4 of 4)

Strict inequality is present in the second and third constraints since

$$\begin{aligned}y_1 &= 3 > 2 \\ -y_1 + y_2 &= 0 > -1.\end{aligned}$$

Thus the second and third components of \mathbf{x} in the primal problem must be zero. Therefore the primal can be recast as **Primal**: Maximize $-3x_1 + 3x_4$ subject to $x_1 \geq 0$, $x_4 \geq 0$ and

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 + x_4 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Thus $x_1 = 5$ and $x_4 = 13$, the maximum of the cost function for the primal is 24 and it occurs at $(x_1, x_2, x_3, x_4) = (5, 0, 0, 13)$.

Duality Theorem

Theorem (Duality Theorem)

One and only one of the following four cases can be true.

- 1. There exist optimal solutions for both the primal and dual problems and the maximum of $\mathbf{c}^T \mathbf{x}$ equals the minimum of $\mathbf{b}^T \mathbf{y}$.*
- 2. There exists no feasible solution to the primal problem and the dual problem has feasible solutions for which the minimum of $\mathbf{b}^T \mathbf{y}$ approaches $-\infty$.*
- 3. There exists no feasible solution to the dual problem and the primal problem has feasible solutions for which the maximum of $\mathbf{c}^T \mathbf{x}$ approaches ∞ .*
- 4. Neither the primal nor the dual problem has a feasible solution.*

Farkas Alternative

Remark: before proving the Duality Theorem we must state a lemma which will be used in the proof.

Lemma (Farkas Alternative)

Exactly one of the following two statements is true. Either

- 1. $A\mathbf{x} \leq \mathbf{b}$ has a solution $\mathbf{x} \geq \mathbf{0}$, or*
- 2. $A^T\mathbf{y} \geq \mathbf{0}$ with $\mathbf{b}^T\mathbf{y} < 0$ has a solution $\mathbf{y} \geq \mathbf{0}$.*

Proof (1 of 8)

Primal: Maximize $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Dual: Minimize $\mathbf{b}^T \mathbf{y}$ subject to $A^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

Assuming there are feasible solutions to each problem then we can re-write the constraint of the dual as $(-A)^T \mathbf{y} \leq -\mathbf{c}$ with $\mathbf{y} \geq \mathbf{0}$. Thus according to the constraint on the primal, the re-written constraint on the dual, and the conclusion of the Weak Duality Theorem the following inequalities hold for $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$.

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ (-A)^T \mathbf{y} &\leq -\mathbf{c} \\ \mathbf{c}^T \mathbf{x} - \mathbf{b}^T \mathbf{y} &\leq \mathbf{0} \end{aligned}$$

Proof (1 of 8)

Primal: Maximize $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Dual: Minimize $\mathbf{b}^T \mathbf{y}$ subject to $A^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

Assuming there are feasible solutions to each problem then we can re-write the constraint of the dual as $(-A)^T \mathbf{y} \leq -\mathbf{c}$ with $\mathbf{y} \geq \mathbf{0}$. Thus according to the constraint on the primal, the re-written constraint on the dual, and the conclusion of the Weak Duality Theorem the following inequalities hold for $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$.

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ (-A)^T \mathbf{y} &\leq -\mathbf{c} \\ \mathbf{c}^T \mathbf{x} - \mathbf{b}^T \mathbf{y} &\leq \mathbf{0} \end{aligned}$$

Remark: if equality holds in the last inequality, then \mathbf{x} and \mathbf{y} are optimal solutions.

Proof (2 of 8)

These inequalities can be written in the block matrix form

$$\begin{bmatrix} A & 0 \\ 0 & -A^T \\ \mathbf{c}^T & -\mathbf{b}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \\ 0 \end{bmatrix}.$$

According to the Farkas Alternative Lemma either this inequality has a solution $\langle \mathbf{x}, \mathbf{y} \rangle \geq \mathbf{0}$ or the alternative

$$\begin{bmatrix} A^T & 0 & \mathbf{c} \\ 0 & -A & -\mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \lambda \end{bmatrix} \geq \mathbf{0} \text{ and } [\mathbf{b}^T \quad -\mathbf{c}^T \quad 0] \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \lambda \end{bmatrix} < 0$$

has a solution $\langle \mathbf{u}, \mathbf{v}, \lambda \rangle \geq \mathbf{0}$.

Proof (3 of 8)

We may decompose this block matrix to derive the following system of inequalities:

$$A^T \mathbf{u} + \lambda \mathbf{c} \geq \mathbf{0}, \quad -A\mathbf{v} - \lambda \mathbf{b} \geq \mathbf{0}, \quad \mathbf{b}^T \mathbf{u} - \mathbf{c}^T \mathbf{v} < 0$$

with $\mathbf{u} \geq \mathbf{0}$, $\mathbf{v} \geq \mathbf{0}$, and $\lambda \geq 0$. If $\lambda > 0$ then this system of inequalities is equivalent to the following system.

$$\begin{aligned} A \begin{pmatrix} 1 \\ \lambda \mathbf{v} \end{pmatrix} &\leq -\mathbf{b} \\ A^T \begin{pmatrix} 1 \\ \lambda \mathbf{u} \end{pmatrix} &\geq -\mathbf{c} \\ -\mathbf{b}^T \begin{pmatrix} 1 \\ \lambda \mathbf{u} \end{pmatrix} &> -\mathbf{c}^T \begin{pmatrix} 1 \\ \lambda \mathbf{v} \end{pmatrix} \end{aligned}$$

Since $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$ the vectors $\frac{1}{\lambda} \mathbf{u} \geq \mathbf{0}$ and $\frac{1}{\lambda} \mathbf{v} \geq \mathbf{0}$ as well.

Proof (4 of 8)

The first two inequalities form a primal problem and its dual.

$$\text{Primal: } A \begin{pmatrix} 1 \\ \lambda \mathbf{v} \end{pmatrix} \leq -\mathbf{b}$$

$$\text{Dual: } A^T \begin{pmatrix} 1 \\ \lambda \mathbf{u} \end{pmatrix} \geq -\mathbf{c}$$

If we apply the Weak Duality Theorem, then it must be the case that $-\mathbf{b}^T \begin{pmatrix} 1 \\ \lambda \mathbf{u} \end{pmatrix} \leq -\mathbf{c}^T \begin{pmatrix} 1 \\ \lambda \mathbf{v} \end{pmatrix}$, contradicting the inequality:

$$-\mathbf{b}^T \begin{pmatrix} 1 \\ \lambda \mathbf{u} \end{pmatrix} > -\mathbf{c}^T \begin{pmatrix} 1 \\ \lambda \mathbf{v} \end{pmatrix}$$

Therefore we know that $\lambda = 0$.

Proof (5 of 8)

Thus the Farkas Alternative simplifies to the following system:

$$A\mathbf{v} \leq \mathbf{0}, \quad A^T\mathbf{u} \geq \mathbf{0}, \quad \text{and} \quad \mathbf{b}^T\mathbf{u} < \mathbf{c}^T\mathbf{v}$$

where $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$. The last inequality implies that $\mathbf{b}^T\mathbf{u} < 0$ or $\mathbf{c}^T\mathbf{v} > 0$. If $\mathbf{b}^T\mathbf{u} < 0$ then the primal problem $A\mathbf{x} \leq \mathbf{b}$ has no feasible solution $\mathbf{x} \geq \mathbf{0}$. To see this note that together the inequalities $\mathbf{x} \geq \mathbf{0}$, $A\mathbf{x} \leq \mathbf{b}$, and $\mathbf{b}^T\mathbf{u} < 0$ imply that

$$\begin{aligned} (A\mathbf{x})^T &\leq \mathbf{b}^T \\ \mathbf{x}^T A^T &\leq \mathbf{b}^T \\ \mathbf{x}^T (A^T\mathbf{u}) &\leq \mathbf{b}^T\mathbf{u} < 0. \end{aligned}$$

However, $\mathbf{x} \geq \mathbf{0}$ and $A^T\mathbf{u} \geq \mathbf{0}$ and thus $\mathbf{x}^T (A^T\mathbf{u}) \geq 0$, a contradiction.

Proof (6 of 8)

We may conclude that if $\mathbf{b}^T \mathbf{u} < 0$ then the primal problem has no feasible solution.

If the dual problem also lacks a feasible solution then we are in the fourth case of the theorem.

If the dual problem possesses a feasible solution \mathbf{y} , then

$$A^T \mathbf{y} + A^T \lambda \mathbf{u} = A^T (\mathbf{y} + \lambda \mathbf{u}) \geq \mathbf{c}.$$

Since $\mathbf{y} + \lambda \mathbf{u} \geq \mathbf{0}$ for all $\lambda \geq 0$, then $\mathbf{y} + \lambda \mathbf{u}$ is a feasible solution to the dual problem, and

$$\lim_{\lambda \rightarrow \infty} \mathbf{b}^T (\mathbf{y} + \lambda \mathbf{u}) = \mathbf{b}^T \mathbf{y} + \lim_{\lambda \rightarrow \infty} (\lambda \mathbf{b}^T \mathbf{u}) = -\infty.$$

Proof (7 of 8)

Returning to the other half of our alternatively, namely $\mathbf{c}^T \mathbf{v} > 0$ and assuming there exists a feasible solution to the dual problem, then we have the following inequalities.

$$\begin{aligned} A^T \mathbf{y} &\geq \mathbf{c} \\ \mathbf{y}^T A &\geq \mathbf{c}^T \\ -\mathbf{y}^T A \mathbf{v} &\geq -\mathbf{c}^T \mathbf{v} < 0 \end{aligned}$$

However, $\mathbf{y} \geq \mathbf{0}$ and $-A \mathbf{v} \geq \mathbf{0}$ and thus $-\mathbf{y}^T A \mathbf{v} \geq 0$, a contradiction.

Therefore the dual problem has no feasible solution.

Proof (8 of 8)

If the primal problem has no feasible solution, then we are once again in the fourth case of the theorem.

If the primal problem has a feasible solution \mathbf{x} , then

$$A\mathbf{x} + A\lambda\mathbf{v} = A(\mathbf{x} + \lambda\mathbf{v}) \leq \mathbf{b}.$$

Since $\mathbf{x} + \lambda\mathbf{v} \geq \mathbf{0}$ for all $\lambda \geq 0$, then $\mathbf{x} + \lambda\mathbf{v}$ is a feasible solution to the primal problem, and

$$\lim_{\lambda \rightarrow \infty} \mathbf{c}^T(\mathbf{x} + \lambda\mathbf{v}) = \mathbf{c}^T\mathbf{x} + \lim_{\lambda \rightarrow \infty} (\lambda\mathbf{c}^T\mathbf{v}) = \infty.$$

Fundamental Theorem of Finance (1 of 2)

Assumptions and background:

- ▶ Experiment has m possible outcomes numbered 1 through m .
- ▶ We can place n wagers (numbered 1 through n) on the outcomes.
- ▶ r_{ij} is the return for a unit bet on wager $i \in \{1, 2, \dots, n\}$ when the outcome of the experiment is $j \in \{1, 2, \dots, m\}$.
- ▶ Vector $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ is called a **betting strategy**. Component x_i is the amount placed on wager i .
- ▶ Return from a betting strategy is $\sum_{i=1}^n x_i r_{ij}$.

Fundamental Theorem of Finance (2 of 2)

Lemma

Exactly one of the following is true: either

- 1. there is a vector of probabilities $\mathbf{p} = \langle p_1, p_2, \dots, p_m \rangle$ for which*

$$\sum_{j=1}^m p_j r_{ij} = 0, \quad \text{for each } i = 1, 2, \dots, n, \text{ or}$$

- 2. there is a betting strategy $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ for which*

$$\sum_{i=1}^n x_i r_{ij} > 0, \quad \text{for each } j = 1, 2, \dots, m.$$

Proof

- ▶ Suppose the first statement is true.
- ▶ Let $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ be a betting strategy.

$$\sum_{j=1}^m p_j \sum_{i=1}^n x_i r_{ji} = \sum_{j=1}^m \sum_{i=1}^n x_i p_j r_{ji} = \sum_{i=1}^n x_i \sum_{j=1}^m p_j r_{ji} = 0$$

- ▶ Since each $p_j \geq 0$ and $\sum_{j=1}^m p_j = 1$ then for some $j \in \{1, 2, \dots, m\}$ it must be the case that

$$\sum_{i=1}^n x_i r_{ji} \leq 0$$

which implies the second statement is false.

Proof

- ▶ Suppose the first statement is true.
- ▶ Let $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ be a betting strategy.

$$\sum_{j=1}^m p_j \sum_{i=1}^n x_i r_{ji} = \sum_{j=1}^m \sum_{i=1}^n x_i p_j r_{ji} = \sum_{i=1}^n x_i \sum_{j=1}^m p_j r_{ji} = 0$$

- ▶ Since each $p_j \geq 0$ and $\sum_{j=1}^m p_j = 1$ then for some $j \in \{1, 2, \dots, m\}$ it must be the case that

$$\sum_{i=1}^n x_i r_{ji} \leq 0$$

which implies the second statement is false.

- ▶ Suppose the second statement is true. If the first statement is also true then the second statement is false.

Interpretation

Considered as an expected value, the first statement of the theorem

$$\sum_{j=1}^m p_j \sum_{i=1}^n x_i r_{ji} = \mathbb{E} \left(\sum_{i=1}^n x_i r_{ji} \right) = 0$$

implies that all betting strategies have an expected return of 0.

Risk Neutral Probability

- ▶ Suppose an person may invest in a collection of stocks S^i for $i = 1, 2, \dots, n$ and save S^0 at the simple interest rate r .
- ▶ After one unit of time the stocks will have values that are described by one of m possible states $\omega_1, \omega_2, \dots, \omega_m$.
- ▶ The probability of achieving state ω_j is p_j .
- ▶ Let $S^i(0)$ be the price of the i th stock at time $t = 0$ and let $S^i(\omega_j)$ be the price of the i th stock at time $t = 1$ under state ω_j .

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If $(1 + r)S^i(0) = \sum_{j=1}^m p_j S^i(\omega_j)$ then $\mathbf{p} = \langle p_1, p_2, \dots, p_m \rangle$ is called a **risk-neutral probability**.

Arbitrage Theorem

Theorem (Arbitrage Theorem)

A risk-neutral probability exists if and only if there is no arbitrage.

We will prove this theorem using the assumptions and notation of the previous slide.

Proof (1 of 6)

- ▶ We may assume $S^0 = 1$ and $p_j > 0$ for future state ω_j .
- ▶ Let y_i be the number of shares of S^i bought or sold at time $t = 0$ and let y_0 be the amount put in savings.
- ▶ Define vectors

$$\begin{aligned}\mathbf{S}(\cdot) &= \langle S^0(\cdot), S^1(\cdot), \dots, S^n(\cdot) \rangle \\ \mathbf{y} &= \langle y_0, y_1, \dots, y_n \rangle.\end{aligned}$$

- ▶ Consider the dual linear program: minimize $(\mathbf{S}(0))^T \mathbf{y}$ subject to the m constraints

$$\begin{aligned}(\mathbf{S}(\omega_1))^T \mathbf{y} &\geq 0 \\ (\mathbf{S}(\omega_2))^T \mathbf{y} &\geq 0 \\ &\vdots \\ (\mathbf{S}(\omega_m))^T \mathbf{y} &\geq 0.\end{aligned}$$

Proof (2 of 6)

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- ▶ For all $M > 1$ then $M\mathbf{y}^*$ is feasible and

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- ▶ There is no type A arbitrage if and only if the minimum of the dual is 0.

Proof (3 of 6)

- ▶ If type B arbitrage exists then the minimum of the objective function is 0 and there exists $j \in \{1, 2, \dots, m\}$ for which strict inequality holds:

$$(\mathbf{S}(\omega_j))^T \mathbf{y} > 0.$$

- ▶ There is no type B arbitrage if

$$\begin{aligned}(\mathbf{S}(0))^T \mathbf{y} &= 0 \\(\mathbf{S}(\omega_1))^T \mathbf{y} &= 0 \\(\mathbf{S}(\omega_2))^T \mathbf{y} &= 0 \\&\vdots \\(\mathbf{S}(\omega_m))^T \mathbf{y} &= 0.\end{aligned}$$

- ▶ The corresponding primal problem has a trivial objective function, that of maximizing $\mathbf{0}^T \mathbf{p} \equiv 0$ for $\mathbf{p} \geq \mathbf{0}$.

Proof (4 of 6)

- ▶ The system of constraints for the primal problem is

$$A\mathbf{p} = \begin{bmatrix} S^0(\omega_1) & S^0(\omega_2) & \cdots & S^0(\omega_m) \\ S^1(\omega_1) & S^1(\omega_2) & \cdots & S^1(\omega_m) \\ \vdots & \vdots & \ddots & \vdots \\ S^n(\omega_1) & S^n(\omega_2) & \cdots & S^n(\omega_m) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix} = \begin{bmatrix} S^0(0) \\ S^1(0) \\ \vdots \\ S^n(0) \end{bmatrix}.$$

- ▶ In the absence of arbitrage, the Duality theorem implies there is an optimal, feasible solution \mathbf{p}^* to the primal problem for which the maximum of the objective function is 0.

Proof (5 of 6)

Consider the first constraint of the primal problem:

$$\langle S^0(\omega_1), S^0(\omega_2), \dots, S^0(\omega_m) \rangle^T \langle p_1^*, p_2^*, \dots, p_m^* \rangle = S^0(0)$$

$$(1+r) \langle 1, 1, \dots, 1 \rangle^T \langle p_1^*, p_2^*, \dots, p_m^* \rangle = 1$$

$$(1+r) \sum_{j=1}^m p_j^* = 1$$

which implies $(1+r)\mathbf{p}^*$ is a risk-neutral probability.

Proof (6 of 6)

To prove the converse:

- ▶ Suppose a risk-neutral probability $\mathbf{p} > \mathbf{0}$ exists.
- ▶ This implies the primal problem is feasible with a maximum value of its objective function equal to 0.
- ▶ By the Duality Theorem there exists an optimal solution \mathbf{y} to the dual problem whose minimum is 0.

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- ▶ Thus there is no type A arbitrage.

Proof (6 of 6)

To prove the converse:

- ▶ Suppose a risk-neutral probability $\mathbf{p} > \mathbf{0}$ exists.
- ▶ This implies the primal problem is feasible with a maximum value of its objective function equal to 0.
- ▶ By the Duality Theorem there exists an optimal solution \mathbf{y} to the dual problem whose minimum is 0.
- ▶ Thus there is no type A arbitrage.
- ▶ Since $\mathbf{p} > \mathbf{0}$ then by the Complementary Slackness principle

$$(\mathbf{S}(\omega_j))^T \mathbf{y} = 0$$

for $j = 1, 2, \dots, m$. Hence there is no type B arbitrage.

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics,
3rd edition, (2012).

author: J. Robert Buchanan

publisher: World Scientific Publishing Co. Pte. Ltd.

address: 27 Warren St., Suite 401–402, Hackensack, NJ
07601

ISBN: 978-9814407441