Arbitrage [a]

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[a]This material is taken from *An Undergraduate Introduction to Financial Mathematics* by J Robert Buchanan, July 2005. All errors of omission or comission are those of J Robert Buchanan.

Introduction

Arbitrage arises from mis-priced financial instruments.

Example 1

- *CostCo sells 100 stamps for* $\$36.75$.
- *USPS sell 100 stamps for* $\$37.00$.

Intuitive Idea

Imagine we will bet on the outcome of an experiment. The **Arbitrage Theorem** states that either the probabilities of the outcomes are such that

- all bets are fair, or
- there is a betting scheme which produces a positive gain independent of the outcome of the experiment.

Odds

The **odds against** an outcome X are related to probabilities of the outcome according to the formula:

$$n : m \text{ against } \Longrightarrow P(X) = \frac{m}{m+n}.$$

The **odds for** an outcome X are related to probabilities of the outcome according to the formula:

$$n : m \text{ in favor } \Longrightarrow P(X) = \frac{n}{m+n}.$$

For a wager of m dollars on a event X with odds against of $n : m$, if X occurs, we win n dollars, otherwise we lose our investment.

Example

Suppose the odds against player A defeating player B in a tennis match are $\frac{3}{2} : 1$ and the odds against player B defeating player A are $\frac{3}{7} : 1$.

$$P(A \text{ wins}) = 0.4 \quad \text{and} \quad P(B \text{ wins}) = 0.7$$

Betting strategy: wager -1 on player A and -2 on player B.

- A wins: lose $3/2$ on the first bet and gain 2 on the second, net gain of $1/2$.
- B wins: gain 1 on the first bet and lose $6/7$ on the second, net gain of $1/7$.

Linear Programming

Let $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ be a column vector in which $x_i \geq 0$ for $i = 1, 2, \ldots, n$.
Let $\mathbf{b} = \langle b_1, b_2, \ldots, b_m \rangle$ be a column vector (with m elements).
Let A be an $m \times n$ matrix.
We say that $\mathbf{x}$ is **feasible** if $A\mathbf{x} = \mathbf{b}$.
If $\mathbf{c}$ is a row vector of n components, then we define

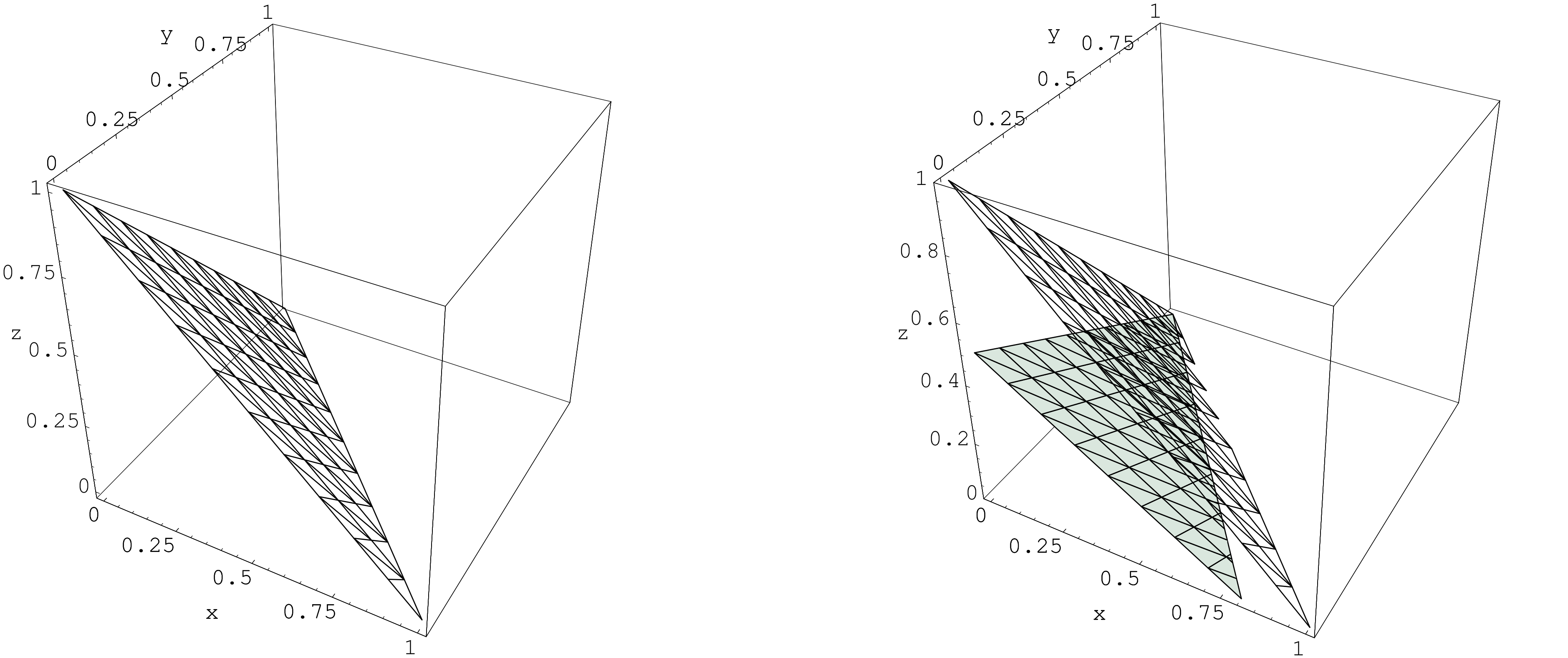
$$\mathbf{c} \cdot \mathbf{x} = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

to be the **cost function.**

We want to minimize the cost function over all possible feasible $\mathbf{x}$.

Example

Minimize $5x_1 + 4x_2 + 8x_3$ subject to $x_1 + x_2 + x_3 = 1$ and $\mathbf{x}$ is feasible.



Slack Variables

Some optimization problems may include **inequality constraints.** If the constraints of the previous example had been $x_1 + x_2 + x_3 \leq 1$ and $\mathbf{x}$ feasible, then the set of points where the minimum must be found would resemble a tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Inequality constraints can be converted to equality constraints by introducing **slack variables.** Thus

$$x_1 + x_2 + x_3 \leq 1 \quad \text{becomes} \quad x_1 + x_2 + x_3 + x_4 = 1,$$

where $x_4 \geq 0$ and "takes up the slack" to produce equality.

What is the minimum of the cost function in the previous example subject to the inequality constraints?

Vector Inequalities

- We say that vector $\mathbf{u}$ is less than (less than or equal to) vector $\mathbf{v}$ if the vectors are both elements of $\mathbb{R}^{1\times k}$ and $u_i < v_i$ ($u_i \leq v_i$) for $i = 1, 2, \ldots, k$.
- Similarly we will say that vector $\mathbf{u}$ is greater than (greater than or equal to) vector $\mathbf{v}$ if $u_i > v_i$ ($u_i \geq v_i$) for $i = 1, 2, \ldots, k$.

These inequalities will be denoted as appropriate $\mathbf{u} < \mathbf{v}$, $\mathbf{u} \leq \mathbf{v}$, $\mathbf{u} > \mathbf{v}$, or $\mathbf{u} \geq \mathbf{v}$.

Dual Problems

For every linear programming problem of the type discussed above, there is an associated problem known as its **dual**. Henceforth the original problem will be known as the **primal**. These paired optimization problems are related in the following ways.

Primal: Minimize $\mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}$ is feasible.

Dual: Maximize $\mathbf{y} \cdot \mathbf{b}$ subject to $\mathbf{y}A \leq \mathbf{c}$.

Observations

Primal: Minimize $\mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}$ is feasible.

Dual: Maximize $\mathbf{y} \cdot \mathbf{b}$ subject to $\mathbf{y}A \leq \mathbf{c}$.

Note:

1. Unknown of dual is row vector $\mathbf{y} \in \mathbb{R}^{1 \times n}$.
2. Vector $\mathbf{b}$ moves from constraint of primal to cost function of dual.
3. Vector $\mathbf{c}$ moves from cost of primal to constraint of the dual.
4. Constraints of dual are inequalities and there are n of them.
5. There is no feasibility constraint (*i.e.* no $\mathbf{y} \geq 0$) in dual.

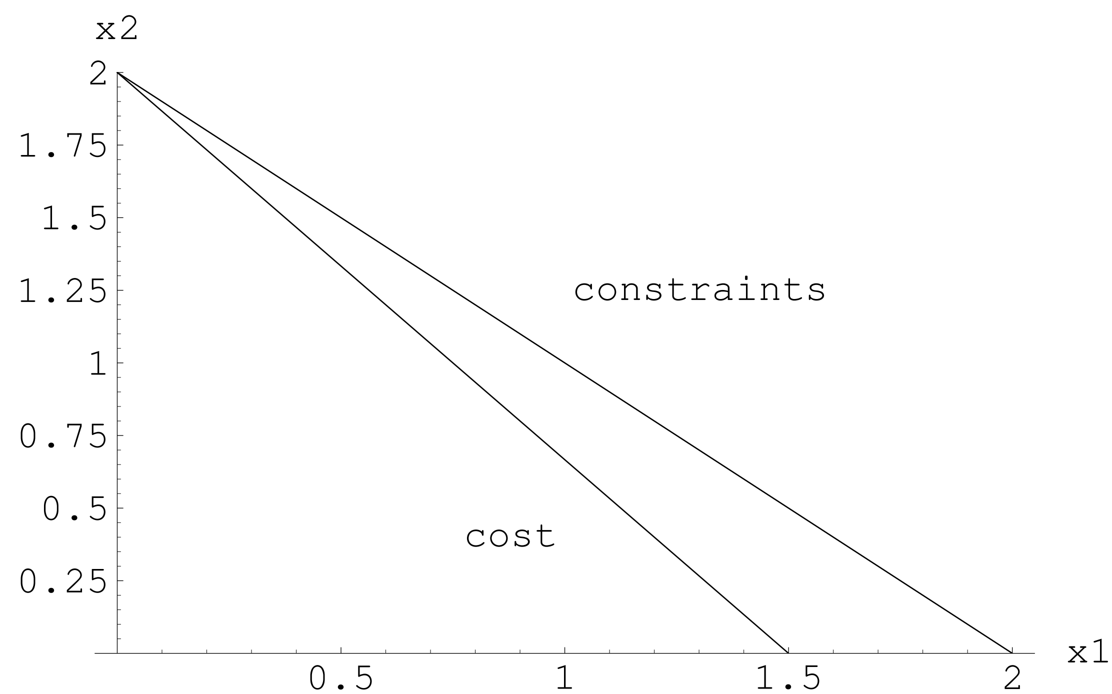
Weak Duality Theorem

Theorem 2 (Weak Duality Theorem) *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are the unknowns of the primal and dual problems respectively, then* $\mathbf{y} \cdot \mathbf{b} \leq \mathbf{c} \cdot \mathbf{x}$*. If* $\mathbf{y} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{x}$ *then these vectors are optimal for their respective problems.*

Example

Primal: Minimize $4x_1 + 3x_2$ subject to $x_1 + x_2 = 2$ and $x_1, x_2 \geq 0$.

Dual: Maximize y_2 subject to $y_2 \leq 3$ and $y_2 \leq 4$.



More on Duality

When $\mathbf{x}$ and $\mathbf{y}$ are optimal for their respective problems then

$$\begin{aligned}
\mathbf{y} \cdot \mathbf{b} = \mathbf{y} A \mathbf{x} &= \mathbf{c} \cdot \mathbf{x} \\
\mathbf{c} \cdot \mathbf{x} - \mathbf{y} \cdot A \mathbf{x} &= 0 \\
\mathbf{c} \cdot \mathbf{x} - (\mathbf{y} A) \cdot \mathbf{x} &= 0 \\
(\mathbf{c} - \mathbf{y} A) \cdot \mathbf{x} &= 0.
\end{aligned}$$

Theorem 3 *Optimality in the primal and dual problems requires either* $x_j = 0$ *or* $(\mathbf{y}A)_j = c_j$ *for* $j = 1, \ldots, n$.

Example

Primal: Minimize $\mathbf{c} \cdot \mathbf{x} = x_1 + 2x_2 + 7x_3 + 3x_4$ subject to $x_i \geq 0$ for $i = 1, 2, 3, 4$ and

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Dual: Maximize $\mathbf{y} \cdot \mathbf{b} = 5y_1 + 3y_2$ subject to

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \leq \begin{bmatrix} 1 & 2 & 7 & 3 \end{bmatrix}$$

Example (cont.)

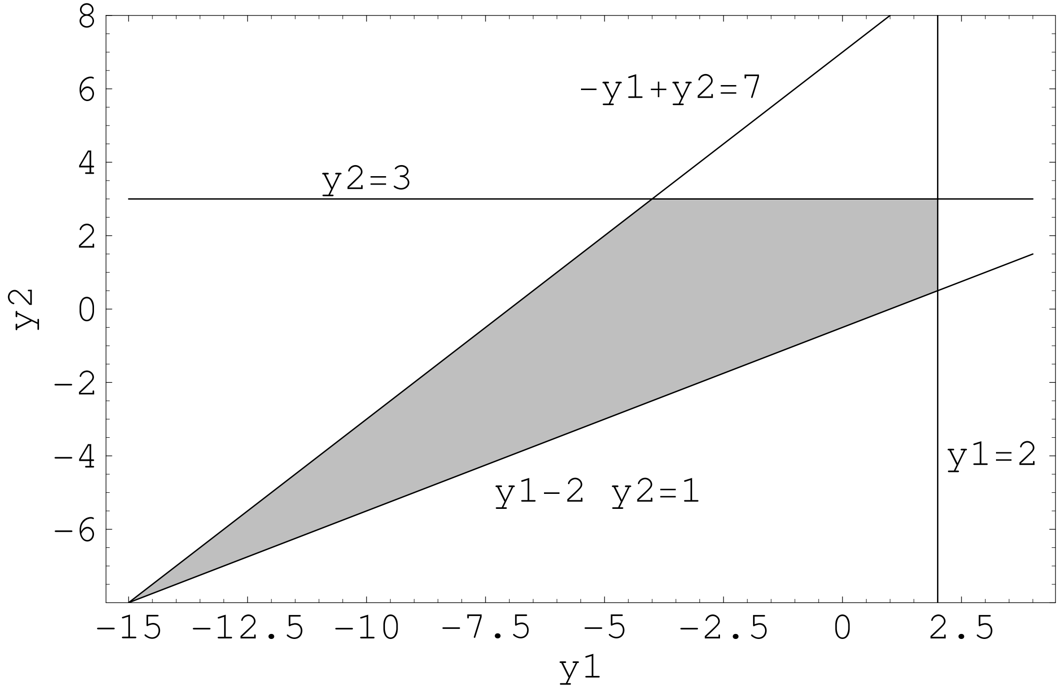
Dual: Maximize $\mathbf{y} \cdot \mathbf{b} = 5y_1 + 3y_2$ subject to

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \leq \begin{bmatrix} 1 & 2 & 7 & 3 \end{bmatrix}$$

Constraints of the dual:

$$\begin{aligned} y_1 - 2y_2 &\leq 1 \\ y_1 &\leq 2 \\ -y_1 + y_2 &\leq 7 \\ y_2 &\leq 3 \end{aligned}$$

Example (cont.)



Maximum of the cost function for the dual occurs at the point with coordinates $(y_1, y_2) = (2, 3)$. Maximum value is 19.

Where does strict equality hold in the dual constraints?

Example (cont.)

First and third components of $\mathbf{x}$ in the primal must be zero. Primal can be restated as

Primal: Minimize $2x_2 + 3x_4$ subject to $x_i \geq 0$ for $i = 2, 4$ and

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

By inspection $x_2 = 5$ and $x_4 = 3$. Minimum of the cost function for primal is 19 and occurs at point with coordinates $(x_1, x_2, x_3, x_4) = (0, 5, 0, 3)$.

Duality Theorem

Theorem 4 (Duality Theorem) *If there is an optimal* $\mathbf{x}$ *in the primal, then there is an optimal* $\mathbf{y}$ *in the dual and the minimum of* $\mathbf{c} \cdot \mathbf{x}$ *equals the maximum of* $\mathbf{y} \cdot \mathbf{b}$*.*

Remarks:

- Dual problem has a dual – namely the primal. Symmetric form of the Duality Theorem: if either the dual or the primal has an optimal solution vector then so does the other and $\mathbf{c} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{b}$.
- If the primal has an inequality constraint $A\mathbf{x} \geq \mathbf{b}$ in place of the equality constraint $A\mathbf{x} = \mathbf{b}$, then the dual possesses the extra constraint $\mathbf{y} \geq 0$.

Remarks (cont.)

Introducing slack variables $\mathbf{x}_s$ in the primal. The primal in matrix form now resembles

$$\begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b}$$

while the dual becomes

$$\mathbf{y} \begin{bmatrix} A & -I \end{bmatrix} \leq \begin{bmatrix} \mathbf{c} & 0 \end{bmatrix}.$$

This implies the two vector inequalities $\mathbf{y}A \leq \mathbf{c}$ and $-\mathbf{y} \leq 0 \Leftrightarrow \mathbf{y} \geq 0$.

Remarks (cont.)

The symmetric form the primal/dual pair can be stated as follows.

Primal: Minimize $\mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} \geq \mathbf{b}$ and $\mathbf{x} \geq 0$.

Dual: Maximize $\mathbf{y} \cdot \mathbf{b}$ subject to $\mathbf{y}A \leq \mathbf{c}$ and $\mathbf{y} \geq 0$.

Arbitrage Theorem

Assumptions and background:

- Experiment has m possible outcomes numbered 1 through m.
- We can place n wagers (numbered 1 through n) on the outcomes.
- r_{ij} is the return for a unit bet on wager i when the outcome of the experiment is j.
- Vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called a **betting strategy.** Component x_i is the amount placed on wager i.
- Return from a betting strategy is $\sum_{i=1}^{n} x_i r_{ij}$.

Fundamental Theorem of Finance

Theorem 5 (Arbitrage Theorem) *Exactly one of the following is true: either*

(1) there is a vector of probabilities $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ *for which*

$$\sum_{j=1}^{m} y_j r_{ij} = 0, \quad \textit{for } i = 1, 2, \ldots, n, \textit{ or}$$

(2) there is a betting strategy $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *for which*

$$\sum_{i=1}^{n} x_i r_{ij} > 0, \quad \textit{for } j = 1, 2, \ldots, m.$$