The Arbitrage Theorem

MATH 472 *Financial Mathematics*

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Objectives

In this lesson we will:

- describe a portfolio of investments as a betting strategy,
- define the concept of risk-neutral probability measure, and
- state and prove the Fundamental Theorem of Asset Pricing (Arbitrage Theorem).

Fundamental Theorem of Finance (1 of 2)

Assumptions and background:

- Experiment has m possible outcomes numbered 1 through m.
- We can place n wagers (numbered 1 through n) on the outcomes.
- r_{ij} is the return for a unit bet on wager $i \in \{1, 2, \ldots, n\}$ when the outcome of the experiment is $j \in \{1, 2, \ldots, m\}$.
- Vector $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ is called a **betting strategy.** Component x_i is the amount placed on wager i.
- Return from a betting strategy is $\sum_{i=1}^{n} x_i r_{ij}$.

Example

An investor can invest in

- a risk-free savings account which will earn 20% simple interest over the next year,
- a motion picture where the rate of return is given in the following table.

	Return	**Probability**
High Success	3.0	0.3
Moderate Success	1.0	0.4
Failure	0.0	0.3

How much should the investor place in each investment?

Solution (1 of 2)

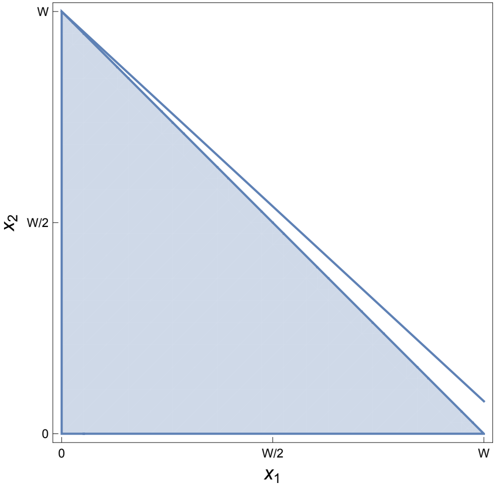
Let the total wealth of the investor be $W > 0$ and suppose x_1 is invested in the savings account and x_2 is invested in the motion picture.

This linear program can be expressed as maximizing

$$1.2x_1 + x_2(0.3 * 3.0 + 0.4 * 1.0 + 0.3 * 0.0) = 1.2x_1 + 1.3x_2$$

subject to $x_1 + x_2 = W$.

Solution (2 of 2)



The investor should place all his/her wealth in the motion picture since it on average returns 30% simple interest.

Fundamental Theorem of Finance (2 of 2)

Lemma

Exactly one of the following is true: either

1. *there is a vector of probabilities* $\mathbf{p} = \langle p_1, p_2, \ldots, p_m \rangle$ *for which*

$$\sum_{j=1}^{m} p_j r_{ij} = \mathbb{E}(r_i) = 0, \text{ for each } i = 1, 2, \ldots, n, \text{ or}$$

2. *there is a betting strategy* $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ *for which*

$$\sum_{i=1}^{n} x_i r_{ij} > 0, \text{ for each } j = 1, 2, \ldots, m.$$

Proof

- Suppose the first statement is true.
- Let $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ be any betting strategy.

$$\sum_{j=1}^{m} p_j \sum_{i=1}^{n} x_i r_{ji} = \sum_{j=1}^{m} \sum_{i=1}^{n} x_i p_j r_{ji} = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} p_j r_{ji} = 0$$

- Since each $p_j \geq 0$ and $\sum_{j=1}^{m} p_j = 1$ then for some $j \in \{1, 2, \ldots, m\}$ it must be the case that

$$\sum_{i=1}^{n} x_i r_{ji} \leq 0$$

which implies the second statement is false.

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  which implies the second statement is false.
- Suppose the second statement is true. If the first statement is also true then the second statement is false.

Interpretation

Considered as an expected value, the first statement of the theorem

$$\sum_{j=1}^{m} p_j \sum_{i=1}^{n} x_i r_{ji} = \mathbb{E}\left(\sum_{i=1}^{n} x_i r_{ji}\right) = 0$$

implies that all betting strategies have an expected return of 0.

Risk Neutral Probability

- Suppose an person may invest in a collection of stocks S^i for $i = 1, 2, \ldots, n$ and save S^0 at the simple interest rate r.
- After one unit of time the stocks will have values that are described by one of m possible states $\omega_1, \omega_2, \ldots, \omega_m$.
- The probability of achieving state ω_j is p_j.
- Let $S^i(0)$ be the price of the ith stock at time $t = 0$ and let $S^i(\omega_j)$ be the price of the ith stock at time $t = 1$ under state ω_j.

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If $(1 + r)S^i(0) = \sum_{j=1}^{m} p_j S^i(\omega_j)$ then $\mathbf{p} = \langle p_1, p_2, \ldots, p_m \rangle$ is called a **risk-neutral probability measure**.

Arbitrage Theorem

Theorem (Arbitrage Theorem)

A risk-neutral probability measure exists if and only if there is no arbitrage.

We will prove this theorem using the assumptions and notation of the previous slide.

Proof (1 of 8)

- We may assume $S^0 = 1$ and $p_j > 0$ for future state ω_j. (If a future state ω_k has corresponding probability $p_k = 0$ then we may ignore it.)
- Let y_i be the number of shares of S^i bought or sold at time $t = 0$ and let y_0 be the amount put in savings.
- Define vectors

$$\begin{aligned}\mathbf{S}(\cdot) &= \langle S^0(\cdot), S^1(\cdot), \ldots, S^n(\cdot)\rangle \\ \mathbf{y} &= \langle y_0, y_1, \ldots, y_n\rangle.\end{aligned}$$

The expression $(\mathbf{S}(0))^T\mathbf{y}$ is the cost function of this investment portfolio.

Proof (2 of 8)

Consider the linear program: minimize $(\mathbf{S}(0))^T\mathbf{y}$ subject to the m constraints

$$\begin{aligned}
(\mathbf{S}(\omega_1))^T\mathbf{y} = S^0(\omega_1)y_0 + S^1(\omega_1)y_1 + \cdots + S^n(\omega_1)y_n &\geq 0 \\
(\mathbf{S}(\omega_2))^T\mathbf{y} = S^0(\omega_2)y_0 + S^1(\omega_2)y_1 + \cdots + S^n(\omega_2)y_n &\geq 0 \\
&\vdots \\
(\mathbf{S}(\omega_m))^T\mathbf{y} = S^0(\omega_m)y_0 + S^1(\omega_m)y_1 + \cdots + S^n(\omega_m)y_n &\geq 0.
\end{aligned}$$

Remarks:

- The constraints imply that whatever the future state of the stock market, the portfolio value is non-negative.
- There are no sign constraints on the components of **y**.

Proof (3 of 8)

- The linear program is feasible since $\mathbf{y} = \mathbf{0}$ satisfies all the constraints.
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- Suppose there exists a feasible solution $\mathbf{y}^*$ for which

$$(\mathbf{S}(0))^T \mathbf{y}^* = c < 0.$$

  If the initial cost of the portfolio is negative, then there is an initial positive cashflow to the investor. The constraints imply there is no probability of loss of this initial cash flow in the future. This is the situation of Type A arbitrage.

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- For all $M > 1$ then $M\,\mathbf{y}^*$ is feasible and

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- There is no type A arbitrage if and only if the minimum of the linear program is 0.

Proof (4 of 8)

- If type B arbitrage exists then the minimum of the cost function is 0 and there exists $j \in \{1, 2, \ldots, m\}$ for which strict inequality holds:

$$(\mathbf{S}(\omega_j))^T \mathbf{y} > 0.$$

- If there is no type B arbitrage, then

$$\begin{aligned}
(\mathbf{S}(0))^T \mathbf{y} &= 0 \\
(\mathbf{S}(\omega_1))^T \mathbf{y} &= 0 \\
(\mathbf{S}(\omega_2))^T \mathbf{y} &= 0 \\
&\vdots \\
(\mathbf{S}(\omega_m))^T \mathbf{y} &= 0.
\end{aligned}$$

Proof (5 of 8)

Let the decision variables in the primal problem be denoted $\mathbf{p} = \langle p_1, p_2, \ldots, p_m \rangle$.

The primal problem is that of maximizing $\langle 0, 0, \ldots, 0 \rangle^T \langle p_1, p_2, \ldots, p_m \rangle = \mathbf{0}^T \mathbf{p} = 0$ subject to

$$A\mathbf{p} = \begin{bmatrix} S^0(\omega_1) & S^0(\omega_2) & \cdots & S^0(\omega_m) \\ S^1(\omega_1) & S^1(\omega_2) & \cdots & S^1(\omega_m) \\ \vdots & \vdots & \ddots & \vdots \\ S^n(\omega_1) & S^n(\omega_2) & \cdots & S^n(\omega_m) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix} = \begin{bmatrix} S^0(0) \\ S^1(0) \\ \vdots \\ S^n(0) \end{bmatrix}.$$

Proof (6 of 8)

- The system of constraints for the primal problem is

$$A\mathbf{p} = \begin{bmatrix} S^0(\omega_1) & S^0(\omega_2) & \cdots & S^0(\omega_m) \\ S^1(\omega_1) & S^1(\omega_2) & \cdots & S^1(\omega_m) \\ \vdots & \vdots & \ddots & \vdots \\ S^n(\omega_1) & S^n(\omega_2) & \cdots & S^n(\omega_m) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix} = \begin{bmatrix} S^0(0) \\ S^1(0) \\ \vdots \\ S^n(0) \end{bmatrix}.$$

- In the absence of arbitrage, the Duality theorem implies there is an optimal, feasible solution $\mathbf{p}^*$ to the primal problem for which the maximum of the objective function is 0.

Proof (7 of 8)

Consider the first constraint of the primal problem:

$$\begin{aligned}
\langle S^0(\omega_1), S^0(\omega_2), \ldots, S^0(\omega_m)\rangle^T \langle p_1^*, p_2^*, \ldots, p_m^*\rangle &= S^0(0) \\
(1+r)\langle 1, 1, \ldots, 1\rangle^T \langle p_1^*, p_2^*, \ldots, p_m^*\rangle &= 1 \\
(1+r)\sum_{j=1}^{m} p_j^* &= 1
\end{aligned}$$

which implies $(1+r)\mathbf{p}^*$ is a risk-neutral probability measure.

Proof (8 of 8)

To prove the converse:

- Suppose a risk-neutral probability measure $\mathbf{p} > \mathbf{0}$ exists.
- This implies the primal problem is feasible with a maximum value of its objective function equal to 0.
- By the Duality Theorem there exists an optimal solution $\mathbf{y}$ to the dual problem whose minimum is 0.

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- Thus there is no type A arbitrage.

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- This implies the primal problem is feasible with a maximum value of its objective function equal to 0.
- By the Duality Theorem there exists an optimal solution $\mathbf{y}$ to the dual problem whose minimum is 0.
- Thus there is no type A arbitrage.
- Since $\mathbf{p} > \mathbf{0}$ then by the Complementary Slackness principle

$$(\mathbf{S}(\omega_j))^T \mathbf{y} = 0$$

for $j = 1, 2, \ldots, m$. Hence there is no type B arbitrage.

Homework

- Read Section 4.4
- Exercises: 16–20

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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