Pricing Options with Binomial Trees

MATH 472 *Financial Mathematics*

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Objectives

In this lesson we will learn:

- a simple discrete framework for pricing options,
- how to calculate risk-neutral probabilities,
- how to price European/American put and call options.

Binomial Model

The binomial model is a discrete approximation to the Black-Scholes initial value problem originally developed by Cox, Ross, and Rubinstein.

Assumptions:

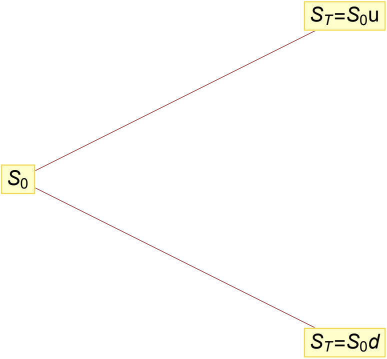
- Strike price of an option is K.
- Exercise time of the option is T.
- Present price of the security is S_0.
- Continuously compounded interest rate is r.
- The growth rate (drift) and volatility of the security are μ and σ respectively.
- Present time is t.

Binomial Lattice

If the value of the stock is S_0 then at $t = T$:

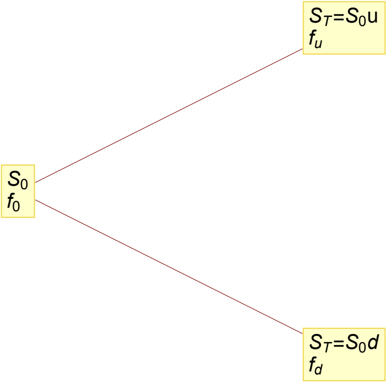
$$S_T = \begin{cases} S_0 u \\ S_0 d \end{cases}$$

where $0 < d < 1 < u$.



Portfolio

Suppose the option (or any financial instrument dependent on the security) has a payoff of f_0 at $t = 0$, f_u at $t = T$ if the security price increases, and f_d at $t = T$ if the security price decreases.



Consider a portfolio which is short one option and long in Δ shares of the security.

Riskless Portfolio (1 of 3)

Determine the value of Δ which makes the payoff of the portfolio the same regardless of whether the security price moves up or down.

At $t = T$ the payoff of the portfolio will be:

$$\text{payoff} = \begin{cases} (\Delta)S_0 u - f_u \\ (\Delta)S_0 d - f_d. \end{cases}$$

Equating the payoffs and solving for Δ yields

$$\begin{aligned} (\Delta)S_0 u - f_u &= (\Delta)S_0 d - f_d \\ \Delta &= \frac{f_u - f_d}{S_0 u - S_0 d}. \end{aligned}$$

Riskless Portfolio (2 of 3)

- Since the payoff of the portfolio is the same independent of the future value of the security, the portfolio is described as **riskless**.
- A riskless investment must earn the risk-free interest rate r.
- The cost of creating the portfolio is $(\Delta)S_0 - f_0$.

$$((\Delta)S_0 - f_0)e^{rT} = (\Delta)S_0 u - f_u = (\Delta)S_0 d - f_d$$

Riskless Portfolio (3 of 3)

We may now determine the arbitrage-free value of the option at $t = 0$.

$$
\begin{aligned}
((\Delta)S_0 - f_0)e^{rT} &= (\Delta)S_0 u - f_u \\
f_0 &= (\Delta)S_0 - ((\Delta)S_0 u - f_u)e^{-rT} \\
&= \left(\frac{f_u - f_d}{S_0 u - S_0 d}\right) S_0 - \left(\left(\frac{f_u - f_d}{S_0 u - S_0 d}\right) S_0 u - f_u\right) e^{-rT} \\
&= (p\, f_u + (1-p) f_d)e^{-rT}
\end{aligned}
$$

where

$$
p = \frac{e^{rT} - d}{u - d}.
$$

Remarks

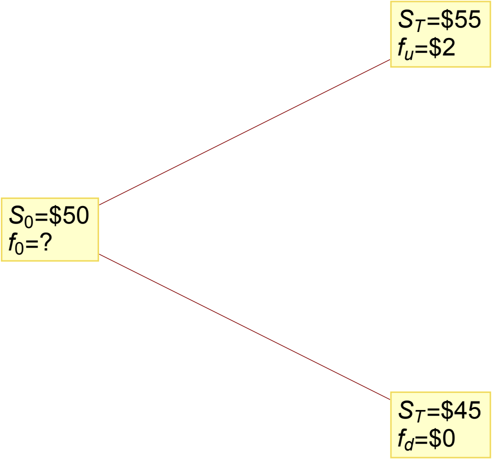
- We have made no assumptions about the probability of an increase/decrease in the value of the security.
- The value of f_0 is independent of the probability of an increase/decrease in the value of the security.
- The quantity $p = \dfrac{e^{rT} - d}{u - d}$ can be *interpreted* as a probability of an increase in the stock price.

$$f_0 = (p\,f_u + (1 - p)f_d)e^{-rT}$$

is the present value of the expected payoff of the option using the probability p.

Example

Suppose the risk-free interest rate is $r = 0.07$ and $T = 0.5$, find the value of the option at $t = 0$ in the following situation.



Solution

$u = 1.1$, $d = 0.9$, $r = 0.07$, $T = 0.5$ which implies

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.07(0.5)} - 0.9}{1.1 - 0.9} = 0.678099.$$

Thus the value of the option is

$$\begin{aligned} f_0 &= (pf_u + (1-p)f_d)e^{-rT} \\ &= (0.678099(2) + (1 - 0.678099)(0))e^{-0.07(0.5)} \\ &= \$1.30955. \end{aligned}$$

Risk Neutral Valuation

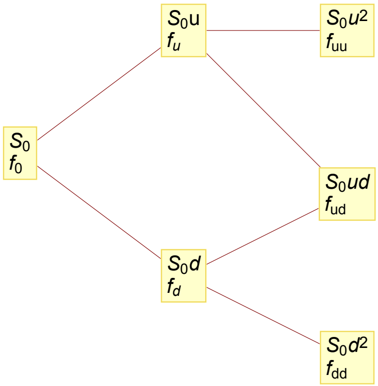
Assuming the probability of the security increasing in value is given by p, what is the expected value of S_T?

$$\begin{aligned}\mathbb{E}[S_T] &= p\,S_0 u + (1-p)S_0 d \\ &= p\,S_0(u-d) + S_0 d \\ &= \left(\frac{e^{rT}-d}{u-d}\right) S_0(u-d) + S_0 d \\ &= S_0 e^{rT}\end{aligned}$$

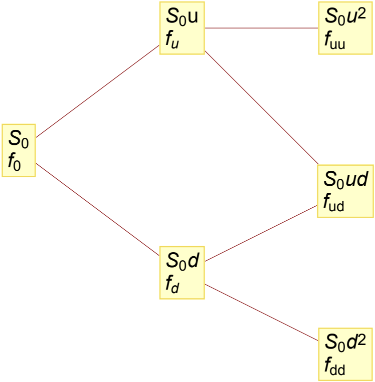
Remark: assuming the probability of an increase in the security value is p is equivalent to assuming the rate of return on the security is the risk-free rate r.

Multi-step Binomial Trees

Suppose the security price is allowed to change every Δt units of time. For example if $\Delta t = T/2$ the binomial tree may resemble the following.



Option Valuation



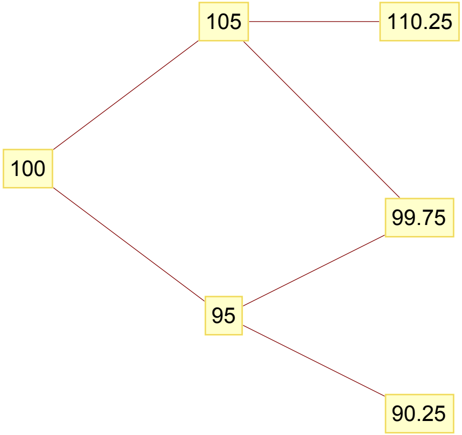
$$
\begin{aligned}
f_u &= (p\, f_{uu} + (1-p) f_{ud}) e^{-r\Delta t} \\
f_d &= (p\, f_{ud} + (1-p) f_{dd}) e^{-r\Delta t} \\
f_0 &= (p\, f_u + (1-p) f_d) e^{-r\Delta t}
\end{aligned}
$$

$$f_0 = (p^2 f_{uu} + 2p(1-p) f_{ud} + (1-p)^2 f_{dd}) e^{-2r\Delta t}$$

Example

Suppose the risk-free interest rate is 3% compounded continuously and the current price of a security is \$100. The stock will increase or decrease in value by 5% each month. Find the value of a two-month European put option on the security with a strike price of \$105. Use a binomial tree with Δt equal to one month.

Solution (1 of 2)



For a 105-strike European put $f_{uu} = 0$,
$f_{ud} = 105 - 99.75 = 5.25$, and $f_{dd} = 105 - 90.25 = 14.75$.

Solution (2 of 2)

$u = 1.05$, $d = 0.95$, $r = 0.03$, $\Delta t = 1/12$ implies

$$
\begin{aligned}
p &= \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.03/12} - 0.95}{1.05 - 0.95} \approx 0.525031 \\
f_0 &= (p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd})e^{-2r\Delta t} \\
&= (p^2(0) + 2p(1-p)(5.25) + (1-p)^2(14.75))e^{-0.03(2/12)} \\
&\approx \$5.9163
\end{aligned}
$$

American Options

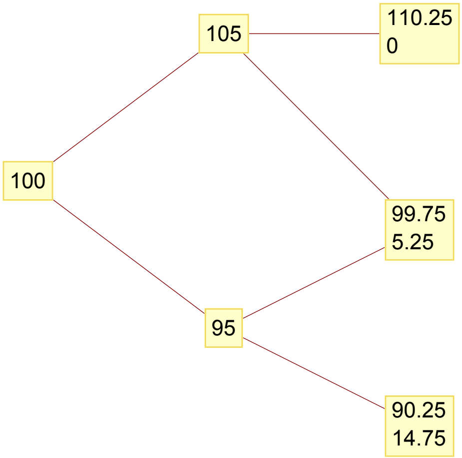
- The binomial tree can be used to price options with American-style exercise.
- The value of the option at the final nodes of the tree is the same as in the case of European-style exercise.
- The value of the option at earlier nodes is the maximum of the **payoff from early exercise** or the **intrinsic value** of the option.

Example: American Put

Suppose the risk-free interest rate is 3% compounded continuously and the current price of a security is \$100. The stock will increase or decrease in value by 5% each month. Find the value of a two-month American put option on the security with a strike price of \$105. Use a binomial tree with Δt equal to one month.

Solution (1 of 5)

The payoffs of the put have already been determined at $t = 2/12$.



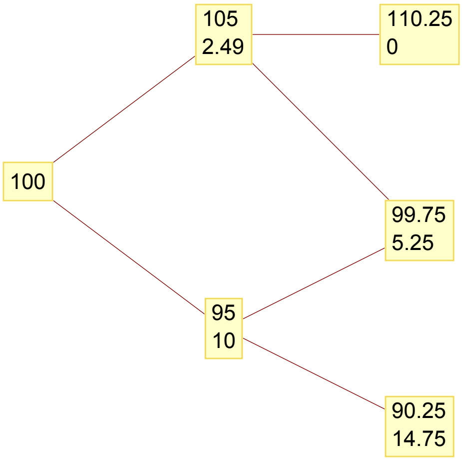
Determine the value of f_u and f_d at $t = 1/12$.

Solution (2 of 5)

$$
\begin{aligned}
f_u &= \max\{(p\,f_{uu} + (1-p)f_{ud})e^{-0.03/12}, 105 - 105\} \\
&= \max\{2.48736, 0\} \\
&= 2.48736 \\
f_d &= \max\{(p\,f_{ud} + (1-p)f_{dd})e^{-0.03/12}, 105 - 95\} \\
&= \max\{9.73783, 10\} \\
&= 10
\end{aligned}
$$

Solution (3 of 5)

The payoffs of the put have already been determined at $t = 1/12$.



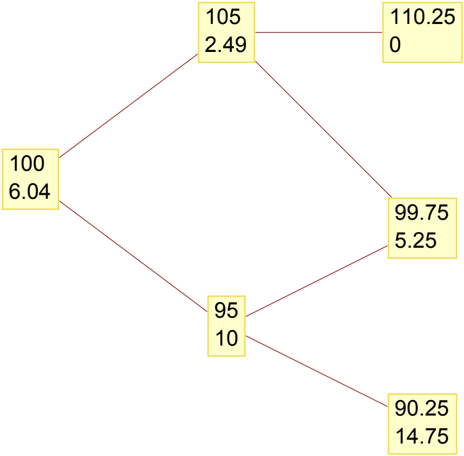
Determine the value of f_0.

Solution (4 of 5)

$$\begin{aligned} f_0 &= \max\{(p\,f_u + (1-p)f_d)e^{-0.03/12}, 105 - 100\} \\ &= \max\{6.04051, 5\} \\ &= 6.04051 \end{aligned}$$

Solution (5 of 5)

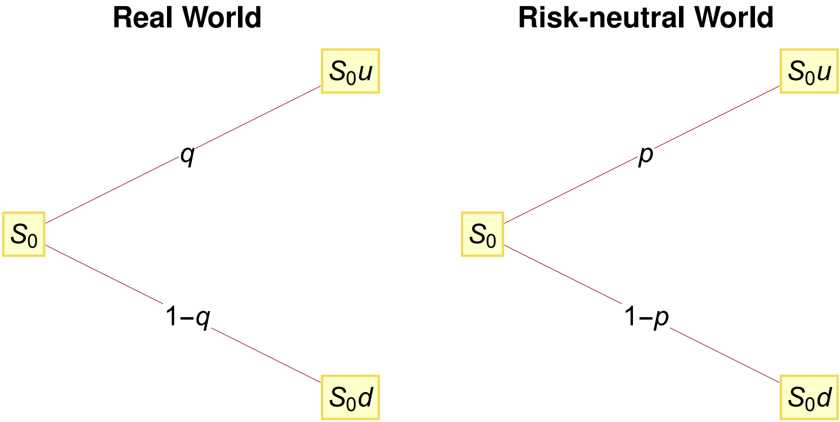
The payoffs of the put at all time steps.



Matching Volatility

- The proportional change in the security price (u and d) should reflect the real-world volatility of the security.
- Suppose the drift and volatility of the security are in the real world μ and σ respectively.
- Let the probability of an increase in the value of the security in the real world be $0 < q < 1$.

Comparing Real and Risk-Neutral Worlds



Probability of Increase in Real World

If the growth rate of the security is μ then a security originally worth S_0 should be worth $S_0 e^{\mu \Delta t}$ after a short time Δt.

Using the binomial tree to determine the expected value of $S_{\Delta t}$ we have,

$$
\begin{aligned}
q\, S_0 u + (1-q) S_0 d &= S_0 e^{\mu \Delta t} \\
q &= \frac{e^{\mu \Delta t} - d}{u - d}.
\end{aligned}
$$

Volatility in the Real World

The variance in the rate of return on the security is assumed to be σ^2, so that in a short interval of time Δt the variance would be $\sigma^2 \Delta t$.

Using the binomial tree to determine the variance in the return we have,

$$\begin{aligned} \sigma^2 \Delta t &= q\,u^2 + (1-q)d^2 - (q\,u + (1-q)d)^2 \\ &= e^{\mu \Delta t}(u+d) - u\,d - e^{2\mu \Delta t}. \end{aligned}$$

If we assume $u\,d = 1$ then we can solve for u and d.

Solving the Equation

$$
\begin{aligned}
u &= \frac{1}{2}e^{-\mu\Delta t}\left(1 + e^{2\mu\Delta t} + \sigma^2\Delta t + \sqrt{(1 + e^{2\mu\Delta t} + \sigma^2\Delta t)^2 - 4e^{2\mu\Delta t}}\right) \\
d &= \frac{1}{2}e^{-\mu\Delta t}\left(1 + e^{2\mu\Delta t} + \sigma^2\Delta t - \sqrt{(1 + e^{2\mu\Delta t} + \sigma^2\Delta t)^2 - 4e^{2\mu\Delta t}}\right)
\end{aligned}
$$

Ignoring terms of $(\Delta t)^2$ or higher powers,

$$
\begin{aligned}
u &\approx e^{\sigma\sqrt{\Delta t}} \\
d &\approx e^{-\sigma\sqrt{\Delta t}}.
\end{aligned}
$$

Homework

- Read Section 7.3
- Exercises: on handout

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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