Pricing Options with Binomial Trees
MATH 472 Financial Mathematics

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In this lesson we will learn:

- a simple discrete framework for pricing options,
- how to calculate risk-neutral probabilities,
- how to price European/American put and call options.
Binomial Model

The binomial model is a discrete approximation to the Black-Scholes initial value problem originally developed by Cox, Ross, and Rubinstein.

**Assumptions:**

- Strike price of an option is $K$.
- Exercise time of the option is $T$.
- Present price of the security is $S_0$.
- Continuously compounded interest rate is $r$.
- The growth rate (drift) and volatility of the security are $\mu$ and $\sigma$ respectively.
- Present time is $t$. 
Binomial Lattice

If the value of the stock is $S_0$ then at $t = T$:

$$S_T = \begin{cases} 
S_0 u \\
S_0 d
\end{cases}$$

where $0 < d < 1 < u$. 

$S_T = S_0 u$

$S_T = S_0 d$

$S_0$
Suppose the option (or any financial instrument dependent on the security) has a payoff of $f_0$ at $t = 0$, $f_u$ at $t = T$ if the security price increases, and $f_d$ at $t = T$ if the security price decreases.

Consider a portfolio which is short one option and long in $\Delta$ shares of the security.
Determine the value of \( \Delta \) which makes the payoff of the portfolio the same regardless of whether the security price moves up or down.

At \( t = T \) the payoff of the portfolio will be:

\[
\text{payoff} = \begin{cases} 
(\Delta)S_0u - f_u \\
(\Delta)S_0d - f_d. 
\end{cases}
\]

Equating the payoffs and solving for \( \Delta \) yields

\[
(\Delta)S_0u - f_u = (\Delta)S_0d - f_d
\]

\[
\Delta = \frac{f_u - f_d}{S_0u - S_0d}.
\]
Since the payoff of the portfolio is the same independent of the future value of the security, the portfolio is described as **riskless**.

A riskless investment must earn the risk-free interest rate $r$.

The cost of creating the portfolio is $(\Delta)S_0 - f_0$.

$$((\Delta)S_0 - f_0)e^{rT} = (\Delta)S_0u - f_u = (\Delta)S_0d - f_d$$
We may now determine the arbitrage-free value of the option at \( t = 0 \).

\[
((\Delta)S_0 - f_0)e^{rT} = (\Delta)S_0u - f_u \\
f_0 = (\Delta)S_0 - ((\Delta)S_0u - f_u)e^{-rT} \\
= \left( \frac{f_u - f_d}{S_0u - S_0d} \right) S_0 - \left( \frac{f_u - f_d}{S_0u - S_0d} \right) S_0u - f_u e^{-rT} \\
= (pf_u + (1 - p)f_d)e^{-rT}
\]

where

\[
p = \frac{e^{rT} - d}{u - d}.
\]
Remarks

- We have made no assumptions about the probability of an increase/decrease in the value of the security.
- The value of $f_0$ is independent of the probability of an increase/decrease in the value of the security.
- The quantity $p = \frac{e^{rT} - d}{u - d}$ can be interpreted as a probability of an increase in the stock price.

$$f_0 = (p f_u + (1 - p) f_d) e^{-rT}$$

is the present value of the expected payoff of the option using the probability $p$. 
Example

Suppose the risk-free interest rate is $r = 0.07$ and $T = 0.5$, find the value of the option at $t = 0$ in the following situation.

$S_0 = $50
$f_0 = ?$

$S_T = $45
$f_d = $0

$S_T = $55
$f_u = $2
Solution

\( u = 1.1, \ d = 0.9, \ r = 0.07, \ T = 0.5 \) which implies

\[
p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.07(0.5)} - 0.9}{1.1 - 0.9} = 0.678099.
\]

Thus the value of the option is

\[
f_0 = (pf_u + (1 - p)f_d)e^{-rT} = (0.678099(2) + (1 - 0.678099)(0))e^{-0.07(0.5)} = $1.30955.
\]
Risk Neutral Valuation

Assuming the probability of the security increasing in value is given by \( p \), what is the expected value of \( S_T \)?

\[
\mathbb{E} [S_T] = p S_0 u + (1 - p) S_0 d \\
= p S_0(u - d) + S_0 d \\
= \left( \frac{e^{rt} - d}{u - d} \right) S_0(u - d) + S_0 d \\
= S_0 e^{rt}
\]

**Remark:** assuming the probability of an increase in the security value is \( p \) is equivalent to assuming the rate of return on the security is the risk-free rate \( r \).
Multi-step Binomial Trees

Suppose the security price is allowed to change every $\Delta t$ units of time. For example if $\Delta t = T/2$ the binomial tree may resemble the following.
Option Valuation

\[ f_u = (p f_{uu} + (1 - p)f_{ud})e^{-r\Delta t} \]

\[ f_d = (p f_{ud} + (1 - p)f_{dd})e^{-r\Delta t} \]

\[ f_0 = (p f_u + (1 - p)f_d)e^{-r\Delta t} \]

\[ f_0 = (p^2 f_{uu} + 2p(1 - p)f_{ud} + (1 - p)^2 f_{dd})e^{-2r\Delta t} \]
Example

Suppose the risk-free interest rate is 3% compounded continuously and the current price of a security is $100. The stock will increase or decrease in value by 5% each month. Find the value of a two-month European put option on the security with a strike price of $105. Use a binomial tree with $\Delta t$ equal to one month.
For a 105-strike European put $f_{uu} = 0$,
$f_{ud} = 105 - 99.75 = 5.25$, and $f_{dd} = 105 - 90.25 = 14.75$. 
Solution (2 of 2)

\[ u = 1.05, \quad d = 0.95, \quad r = 0.03, \quad \Delta t = 1/12 \] implies

\[ p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.03/12} - 0.95}{1.05 - 0.95} \approx 0.525031 \]

\[ f_0 = (p^2 f_{uu} + 2p(1 - p)f_{ud} + (1 - p)^2 f_{dd})e^{-2r\Delta t} \]
\[ = (p^2(0) + 2p(1 - p)(5.25) + (1 - p)^2(14.75))e^{-0.03(2/12)} \]
\[ \approx \$5.9163 \]
American Options

- The binomial tree can be used to price options with American-style exercise.
- The value of the option at the final nodes of the tree is the same as in the case of European-style exercise.
- The value of the option at earlier nodes is the maximum of the payoff from early exercise or the intrinsic value of the option.
Example: American Put

Suppose the risk-free interest rate is 3% compounded continuously and the current price of a security is $100. The stock will increase or decrease in value by 5% each month. Find the value of a two-month American put option on the security with a strike price of $105. Use a binomial tree with $\Delta t$ equal to one month.
Solution (1 of 5)

The payoffs of the put have already been determined at \( t = 2/12 \).

Determine the value of \( f_u \) and \( f_d \) at \( t = 1/12 \).
Solution (2 of 5)

\[ f_u = \max\{(p f_{uu} + (1 - p)f_{ud})e^{-0.03/12}, 105 - 105\} \]
\[ = \max\{2.48736, 0\} \]
\[ = 2.48736 \]

\[ f_d = \max\{(p f_{ud} + (1 - p)f_{dd})e^{-0.03/12}, 105 - 95\} \]
\[ = \max\{9.73783, 10\} \]
\[ = 10 \]
Solution (3 of 5)

The payoffs of the put have already been determined at \( t = 1/12 \).

Determine the value of \( f_0 \).
\[ f_0 = \max\left\{ (p f_u + (1 - p) f_d) e^{-0.03/12}, 105 - 100 \right\} \]
\[ = \max\{6.04051, 5\} \]
\[ = 6.04051 \]
Solution (5 of 5)

The payoffs of the put at all time steps.

- 100
- 6.04
- 95
- 10
- 105
- 2.49
- 90.25
- 14.75
- 99.75
- 5.25
- 110.25
- 0
- 90.25
- 14.75
- 99.75
- 5.25
- 110.25
- 0
Matching Volatility

- The proportional change in the security price ($u$ and $d$) should reflect the real-world volatility of the security.
- Suppose the drift and volatility of the security are in the real world $\mu$ and $\sigma$ respectively.
- Let the probability of an increase in the value of the security in the real world be $0 < q < 1$. 
Comparing Real and Risk-Neutral Worlds

Real World

Risk-neutral World

$q \quad S_0 S_0 u

1-q \quad S_0 d

p \quad S_0 S_0 u

1-p \quad S_0 d
Probability of Increase in Real World

If the growth rate of the security is $\mu$ then a security originally worth $S_0$ should be worth $S_0 e^{\mu \Delta t}$ after a short time $\Delta t$.

Using the binomial tree to determine the expected value of $S_{\Delta t}$ we have,

$$q S_0 u + (1 - q) S_0 d = S_0 e^{\mu \Delta t}$$

$$q = \frac{e^{\mu \Delta t} - d}{u - d}.$$
Volatility in the Real World

The variance in the rate of return on the security is assumed to be $\sigma^2$, so that in a short interval of time $\Delta t$ the variance would be $\sigma^2 \Delta t$.

Using the binomial tree to determine the variance in the return we have,

$$
\sigma^2 \Delta t = q u^2 + (1 - q) d^2 - (q u + (1 - q) d)^2
$$

$$
= e^{\mu \Delta t} (u + d) - u d - e^{2 \mu \Delta t}.
$$

If we assume $u d = 1$ then we can solve for $u$ and $d$. 
Solving the Equation

\[ u = \frac{1}{2} e^{-\mu \Delta t} \left( 1 + e^{2\mu \Delta t} + \sigma^2 \Delta t + \sqrt{(1 + e^{2\mu \Delta t} + \sigma^2 \Delta t)^2 - 4e^{2\mu \Delta t}} \right) \]

\[ d = \frac{1}{2} e^{-\mu \Delta t} \left( 1 + e^{2\mu \Delta t} + \sigma^2 \Delta t - \sqrt{(1 + e^{2\mu \Delta t} + \sigma^2 \Delta t)^2 - 4e^{2\mu \Delta t}} \right) \]

Ignoring terms of \((\Delta t)^2\) or higher powers,

\[ u \approx e^{\sigma \sqrt{\Delta t}} \]

\[ d \approx e^{-\sigma \sqrt{\Delta t}}. \]
Homework

- Read Section 7.3
- Exercises: on handout
These slides are adapted from the textbook,


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