Initial Value Problem for the European Call

The main objective of this lesson is solving the Black-Scholes initial boundary value problem.

For \((S, t)\) in \([0, \infty) \times [0, T] \),

\[
\begin{align*}
    rF &= F_t + rSF_S + \frac{1}{2} \sigma^2 S^2 F_{SS} \\
    F(S, T) &= (S(T) - K)^+ \quad \text{for} \ S > 0, \\
    F(0, t) &= 0 \quad \text{for} \ 0 \leq t < T, \\
    F(S, t) &= S - Ke^{-r(T-t)} \quad \text{as} \ S \to \infty.
\end{align*}
\]
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\end{align*}
\]

We will solve this system of equations using Fourier Transforms.
Fourier Transform of a Function

Definition
If \( f : \mathbb{R} \rightarrow \mathbb{R} \) then the **Fourier Transform** of \( f \) is

\[
\mathcal{F}\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, dx,
\]

where \( i = \sqrt{-1} \) and \( \omega \) is a parameter. The Fourier Transform exists only if the improper integral converges.
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The Fourier Transform of \( f \) will exist when

- \( f \) and \( f' \) are piecewise continuous on every interval of the form \([-M, M]\) for arbitrary \( M > 0 \), and
- \( \int_{-\infty}^{\infty} |f(x)| \, dx \) converges.
When working with complex-valued exponentials, the Euler Identity may be useful:

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]
When working with complex-valued exponentials, the **Euler Identity** may be useful:

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

Example

Find the Fourier Transform of the piecewise-defined function

\[ f(x) = \begin{cases} 
1/2 & \text{if } |x| \leq 1, \\
0 & \text{otherwise.}
\end{cases} \]
Solution

\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \]

\[ = \int_{-1}^{1} \frac{1}{2} e^{-i\omega x} \, dx \]

\[ = \frac{-1}{2i\omega} e^{-i\omega x} \bigg|_{-1}^{1} \]

\[ = \frac{-1}{2i\omega} \left( e^{-i\omega} - e^{i\omega} \right) \]

\[ = \frac{1}{\omega} \left( \frac{e^{i\omega} - e^{-i\omega}}{2i} \right) \]

\[ = \frac{1}{\omega} \left( \frac{\cos \omega + i \sin \omega - \cos \omega + i \sin \omega}{2i} \right) \]

\[ \hat{f}(\omega) = \frac{\sin \omega}{\omega} \]
\[ f(x) = \begin{cases} \frac{1}{2} & \text{if } |x| \leq 1, \\ 0 & \text{otherwise}. \end{cases} \]

\[ \hat{f}(\omega) = \frac{\sin \omega}{\omega} \]
Suppose the Fourier Transform of $f$ exists and that $f'$ exists, find $\mathcal{F}\{f'(x)\}$.

**Hint**: use integration by parts.
Solution

Applying integration by parts with

\[ u = e^{-i\omega x} \quad v = f(x) \]

\[ du = -i\omega e^{-i\omega x} \, dx \quad dv = f'(x) \, dx \]

yields

\[
\mathcal{F} \{ f'(x) \} = \int_{-\infty}^{\infty} f'(x)e^{-i\omega x} \, dx
\]

\[
= f(x)e^{-i\omega x} \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega)e^{-i\omega x} \, dx
\]

\[
= i\omega \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, dx
\]

\[
= i\omega \hat{f}(\omega).
\]
Theorem
If \( f(x), f'(x), \ldots, f^{(n-1)}(x) \) are all Fourier transformable and if \( f^{(n)}(x) \) exists (where \( n \in \mathbb{N} \)) then \( \mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n \hat{f}(\omega) \).
Proof

The previous example demonstrates the result is true for \( n = 1 \). Suppose the result is true for \( n = k \geq 1 \). By definition

\[
\mathcal{F} \left\{ f^{(k+1)}(x) \right\} = \int_{-\infty}^{\infty} f^{(k+1)}(x) e^{-i\omega x} \, dx
\]

\[
= f^{(k)}(x) e^{-i\omega x} \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f^{(k)}(x) (-i\omega) e^{-i\omega x} \, dx
\]

\[
= (i\omega) \int_{-\infty}^{\infty} f^{(k)}(x) e^{-i\omega x} \, dx
\]

\[
= (i\omega)(i\omega)^{k} \hat{f}(\omega)
\]

\[
= (i\omega)^{k+1} \hat{f}(\omega).
\]

The result follows by induction on \( k \).
Fourier Convolution

Definition
The **Fourier Convolution** of two functions $f$ and $g$ is

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - z)g(z) \, dz,$$

provided the improper integral converges.
Fourier Convolution

**Definition**
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**Theorem**

$\mathcal{F}\{(f * g)(x)\} = \hat{f}(\omega)\hat{g}(\omega)$, in other words the Fourier Transform of the Fourier Convolution of $f$ and $g$ is the product of the Fourier Transforms of $f$ and $g$. 
Proof (1 of 2)

\[ \mathcal{F}\{(f \ast g)(x)\} = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x - z)g(z) \, dz \right] e^{-i\omega x} \, dx \]

\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x - z)g(z)e^{-i\omega x} \, dz \right] dx \]

\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x - z)g(z)e^{-i\omega x} \, dx \right] dz \]

\[ = \int_{-\infty}^{\infty} g(z) \left[ \int_{-\infty}^{\infty} f(x - z)e^{-i\omega x} \, dx \right] dz \]
Proof (2 of 2)

So far,

\[
\mathcal{F}\{(f \ast g)(x)\} = \int_{-\infty}^{\infty} g(z) \left[ \int_{-\infty}^{\infty} f(x - z)e^{-i\omega x} \, dx \right] \, dz
\]

\[
= \int_{-\infty}^{\infty} g(z) \left[ \int_{-\infty}^{\infty} f(u)e^{-i\omega(u+z)} \, du \right] \, dz
\]

\[
= \int_{-\infty}^{\infty} g(z)e^{-i\omega z} \left[ \int_{-\infty}^{\infty} f(u)e^{-i\omega u} \, du \right] \, dz
\]

\[
= \hat{f}(\omega) \int_{-\infty}^{\infty} g(z)e^{-i\omega z} \, dz
\]

\[
= \hat{f}(\omega) \hat{g}(\omega).
\]
Example

Let

\[ f(x) = \cos x \]
\[ g(x) = \begin{cases} 1/2 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise}. \end{cases} \]

and find \((f \ast g)(x)\).
Solution

\[(f \ast g)(x) = \int_{-\infty}^{\infty} f(x - z) g(z) \, dz\]

\[= \int_{-1}^{1} \frac{1}{2} \cos(x - z) \, dz\]

\[= \frac{1}{2} \int_{-1}^{1} (\cos x \cos z + \sin x \sin z) \, dz\]

\[= \frac{1}{2} \cos x \int_{-1}^{1} \cos z \, dz\]

\[= \sin(1) \cos x\]
Let

\[ f(x) = \begin{cases} 
1 - x & \text{if } 0 \leq x < 1, \\
0 & \text{otherwise.}
\end{cases} \]

\[ g(x) = \begin{cases} 
e^{-x} & \text{if } x \geq 0, \\
0 & \text{otherwise.}
\end{cases} \]

and show \( \mathcal{F}\{(f \ast g)(x)\} = \hat{f}(\omega)\hat{g}(\omega) \).
Solution (1 of 4)

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, dx
\]

\[
= \int_{0}^{1} (1 - x)e^{-i\omega x} \, dx
\]

\[
= \frac{-1}{i\omega}(1 - x)e^{-i\omega x}\Bigg|_{0}^{1} - \int_{0}^{1} \frac{1}{i\omega}e^{-i\omega x} \, dx
\]

\[
= \frac{1}{\omega^2} \left( 1 - e^{-i\omega} \right) + \frac{1}{i\omega}
\]
\[
\hat{g}(\omega) = \int_{-\infty}^{\infty} g(x) e^{-i\omega x} \, dx \\
= \int_{0}^{\infty} e^{-x} e^{-i\omega x} \, dx \\
= \left. \frac{-1}{1 + i\omega} e^{-(1+i\omega)x} \right|_{0}^{\infty} \\
= \frac{1}{1 + i\omega} \\
\]
Solution (2 of 4)

\[ \hat{g}(\omega) = \int_{-\infty}^{\infty} g(x) e^{-i\omega x} \, dx \]
\[ = \int_{0}^{\infty} e^{-x} e^{-i\omega x} \, dx \]
\[ = \left. \frac{-1}{1 + i\omega} e^{-(1+i\omega)x} \right|_{0}^{\infty} \]
\[ = \frac{1}{1 + i\omega} \]

Thus

\[ \hat{f}(\omega) \hat{g}(\omega) = \frac{1 - e^{-i\omega} - i\omega}{\omega^2(1 + i\omega)}. \]
Solution (3 of 4)

Now find the convolution.

\[(f \ast g)(x) = \int_{-\infty}^{\infty} f(z)g(x - z) \, dz\]

\[= \int_{0}^{\min(1, x)} (1 - z)e^{-(x-z)} \, dz\]

\[= (2 - z)e^{-(x-z)} \bigg|_{0}^{\min(1, x)}\]

\[h(x) = \begin{cases} 
0 & \text{if } x < 0, \\
2 - x - 2e^{-x} & \text{if } 0 \leq x < 1, \\
(e - 2)e^{-x} & \text{if } x \geq 1.
\end{cases}\]
Now find the Fourier transform of the convolution.

\[ \hat{h}(\omega) = \int_{-\infty}^{\infty} h(x) e^{-i\omega x} \, dx \]

\[ = \int_{0}^{1} (2 - x - 2e^{-x}) e^{-i\omega x} \, dx + \int_{1}^{\infty} (e - 2) e^{-x} e^{-i\omega x} \, dx \]

\[ = e^{-1-i\omega} \left( i(e + (e - 2)\omega^2) - (i + \omega) e^{1+i\omega} \right) + \frac{(e - 2)e^{-1-i\omega}}{1 - i\omega} \]

\[ = \frac{1 - e^{-i\omega} - i\omega}{\omega^2(1 + i\omega)} \]
Inverse Fourier Transform

Definition
The **inverse Fourier Transform** of \( \hat{f}(\omega) \) is denoted \( \mathcal{F}^{-1}\{\hat{f}(\omega)\} \) and is given by

\[
\mathcal{F}^{-1}\{\hat{f}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega.
\]
Example

Find the inverse Fourier Transform of $e^{-|\omega|}$. 
Solution

\[
\mathcal{F}^{-1}\left\{ e^{-|\omega|} \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\omega|} e^{i\omega x} \, d\omega \\
= \frac{1}{2\pi} \int_{-\infty}^{0} e^{(1+ix)\omega} \, d\omega + \frac{1}{2\pi} \int_{0}^{\infty} e^{(-1+ix)\omega} \, d\omega \\
= \left. \frac{1}{2\pi} \frac{1}{1+ix} e^{(1+ix)\omega} \right|_{-\infty}^{0} + \left. \frac{1}{2\pi} \frac{1}{-1+ix} e^{(-1+ix)\omega} \right|_{0}^{\infty} \\
= \frac{1}{2\pi(1+ix)} + \frac{1}{2\pi(1-ix)} \\
f(x) = \frac{1}{\pi(1+x^2)}
\]
\[ \hat{f}(\omega) = e^{-|\omega|} \]
\[ f(x) = \frac{1}{\pi(1 + x^2)} \]
We will use the Fourier Transform and its inverse to solve the Black-Scholes PDE once we have performed a suitable change of variables on the PDE.

Define the new variables $x$, $\tau$, and $v$ as

\[
x = \ln \frac{S}{K},
\]
\[
\tau = \frac{\sigma^2}{2} (T - t),
\]
\[
v(x, \tau) = \frac{1}{K} F(S, t)
\]

and calculate $F_t$, $F_S$, and $F_{SS}$. 

Change of Variables

By the Chain Rule:

\[
F_t = K v_x x_t + K v_{\tau} \tau_t = -\frac{K \sigma^2}{2} v_{\tau}
\]

\[
F_S = K v_x x_S + K v_{\tau} \tau_S = e^{-x} v_x
\]

\[
F_{SS} = -e^{-x} v_x x_S + e^{-x} v_{xx} x_S = \frac{e^{-2x}}{K} (v_{xx} - v_x).
\]
Change of Variables

By the Chain Rule:

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\[ F_{SS} = -e^{-x} v_x x_S + e^{-x} v_{xx} x_S = \frac{e^{-2x}}{K} (v_{xx} - v_x). \]

Substituting into the Black-Scholes PDE:

\[ r F = F_t + r S F_S + \frac{1}{2} \sigma^2 S^2 F_{SS} \]
Change of Variables

By the Chain Rule:

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F_t = K v_x x_t + K v_{\tau t} = -\frac{K \sigma^2}{2} v_{\tau}
\]

\[
F_S = K v_x x_S + K v_{\tau S} = e^{-x} v_x
\]

\[
F_{SS} = -e^{-x} v_x x_S + e^{-x} v_{xx} x_S = \frac{e^{-2x}}{K} (v_{xx} - v_x).
\]

Substituting into the Black-Scholes PDE:

\[
r F = F_t + r S F_S + \frac{1}{2} \sigma^2 S^2 F_{SS}
\]

\[
v_{\tau} = v_{xx} + (k - 1) v_x - k v
\]

where \( k = 2r / \sigma^2 \).
Under the change of variables the final condition becomes an initial condition.

\[
F(S, T) = (S(T) - K)^+ \\
Kv(x, 0) = (Ke^x - K)^+ \\
v(x, 0) = (e^x - 1)^+
\]

becomes an initial condition.
Under the change of variables the final condition

\[ \begin{align*}
F(S, T) &= (S(T) - K)^+ \\
Kv(x, 0) &= (Ke^x - K)^+ \\
v(x, 0) &= (e^x - 1)^+
\end{align*} \]

becomes an initial condition.

The boundary condition

\[ \begin{align*}
F(0, t) &= \lim_{S \to 0^+} F(S, t) \\
0 &= \lim_{x \to -\infty} K v(x, \tau) \\
0 &= \lim_{x \to -\infty} v(x, \tau).
\end{align*} \]

The boundary at \( S = 0 \) has moved to a boundary as \( x \to -\infty \).
The boundary condition

\[
\lim_{S \to \infty} F(S, t) = S - Ke^{-r(T-t)}
\]

\[
\lim_{x \to \infty} K v(x, \tau) = Ke^x - Ke^{-r(T-[T-2\tau/\sigma^2])}
\]

\[
\lim_{x \to \infty} v(x, \tau) = e^x - e^{-k\tau}.
\]
The initial value problem in the new variables is

\[ \nu_\tau = \nu_{xx} + (k - 1) \nu_x - k \nu \quad \text{for} \quad x \in (-\infty, \infty), \; \tau \in (0, \frac{T\sigma^2}{2}) \]

\[ \nu(x, 0) = (e^x - 1)^+ \quad \text{for} \quad x \in (-\infty, \infty) \]

\[ \nu(x, \tau) \to 0 \quad \text{as} \quad x \to -\infty \quad \text{and} \]

\[ \nu(x, \tau) \to e^x - e^{-k\tau} \quad \text{as} \quad x \to \infty, \; \tau \in (0, \frac{T\sigma^2}{2}). \]
Another Change of Variables

Suppose $\alpha$ and $\beta$ are constants and

$$v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau).$$

Find $v_x, v_{xx}$, and $v_\tau$ in terms of $u_x, u_{xx}$, and $u_\tau$. 
Another Change of Variables

Suppose $\alpha$ and $\beta$ are constants and

$$v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau).$$

Find $v_x$, $v_{xx}$, and $v_\tau$ in terms of $u_x$, $u_{xx}$, and $u_\tau$.

$$v_x = e^{\alpha x + \beta \tau} \left( \alpha u(x, \tau) + u_x \right)$$

$$v_{xx} = e^{\alpha x + \beta \tau} \left( \alpha^2 u(x, \tau) + 2\alpha u_x + u_{xx} \right)$$

$$v_\tau = e^{\alpha x + \beta \tau} \left( \beta u(x, \tau) + u_\tau \right)$$
Substituting into the PDE

If we substitute function $u$ in place of function $v$ in the IBVP, we obtain:

$$v_\tau = v_{xx} + (k - 1)v_x - kv$$
$$u_\tau = (\alpha^2 + (k - 1)\alpha - k - \beta)u + (2\alpha + k - 1)u_x + u_{xx}$$
Substituting into the PDE

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\begin{align*}
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\end{align*}
$$

If we choose $\alpha$ and $\beta$ so that

$$
\begin{align*}
    0 &= \alpha^2 + (k - 1)\alpha - k - \beta \\
    0 &= 2\alpha + k - 1
\end{align*}
$$

then the first two terms on the right-hand side of the equation for $u$ vanish.
Substituting into the PDE

If we substitute function $u$ in place of function $v$ in the IBVP, we obtain:

\[
\begin{align*}
v_{\tau} &= v_{xx} + (k - 1)v_x - kv \\
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\]

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0 &= 2\alpha + k - 1
\end{align*}
\]

then the first two terms on the right-hand side of the equation for $u$ vanish.

Find $\alpha$ and $\beta$. 

Heat Equation

If we let

\[ \alpha = \frac{1 - k}{2} \]
\[ \beta = -\frac{(1 + k)^2}{4} \]

then the PDE:

\[ u_\tau = (\alpha^2 + (k - 1)\alpha - k - \beta)u + (2\alpha + k - 1)u_x + u_{xx} \]

can be written as

\[ u_\tau = u_{xx} \]

which is known as the Heat Equation.
Side Conditions

If $\alpha = \frac{1 - k}{2}$ and $\beta = -\frac{(k + 1)^2}{4}$, then the initial condition becomes:

\[
\begin{align*}
v(x, 0) &= (e^x - 1)^+ \\
u(x, 0) &= (e^{(k+1)x/2} - e^{(k-1)x/2})^+.
\end{align*}
\]
Side Conditions

If \( \alpha = \frac{1 - k}{2} \) and \( \beta = -\frac{(k + 1)^2}{4} \), then the initial condition becomes:

\[
\begin{align*}
\nu(x, 0) &= (e^x - 1)^+ \\
\mu(x, 0) &= (e^{(k+1)x/2} - e^{(k-1)x/2})^+.
\end{align*}
\]

The boundary conditions become

\[
\begin{align*}
\lim_{x \to -\infty} \nu(x, \tau) &= 0 \\
\lim_{x \to -\infty} \mu(x, \tau) &= 0
\end{align*}
\]

and

\[
\begin{align*}
\lim_{x \to \infty} \nu(x, \tau) &= e^x - e^{-k\tau} \\
\lim_{x \to \infty} \mu(x, \tau) &= e^{(k+1)x/2} [x + (k+1)\tau/2] - e^{(k-1)x/2} [x + (k-1)\tau/2].
\end{align*}
\]
The final form of the Black-Scholes IBVP can be summarized as follows.

\[
\begin{align*}
    u_\tau &= u_{xx} \quad \text{for} \ x \in (-\infty, \infty) \ \text{and} \ \tau \in (0, T\sigma^2/2) \\
    u(x, 0) &= (e^{(k+1)x/2} - e^{(k-1)x/2})^+ \quad \text{for} \ x \in (-\infty, \infty) \\
    u(x, \tau) &\to 0 \quad \text{as} \ x \to -\infty \ \text{for} \ \tau \in (0, T\sigma^2/2) \\
    u(x, \tau) &\to e^{(k+1)x/2}[x+(k+1)\tau/2] - e^{(k-1)x/2}[x+(k-1)\tau/2] \\
        &\quad \quad \text{as} \ x \to \infty \ \text{for} \ \tau \in (0, T\sigma^2/2)
\end{align*}
\]
We now turn to the Fourier Transform to solve the IBVP.

\[
\begin{align*}
    u_\tau &= u_{xx} \\
    \mathcal{F}\{u_\tau\} &= \mathcal{F}\{u_{xx}\} \\
    \frac{d\hat{u}}{d\tau} &= -\omega^2 \hat{u} \\
    \hat{u}(\omega, \tau) &= D e^{-\omega^2 \tau}
\end{align*}
\]

where \( D = \hat{f}(\omega) \) is the Fourier Transform of the initial condition.
Inverse Fourier Transforming the Solution

Recall the Fourier Convolution and the Fourier Transform of the Fourier Convolution.

\[ \mathcal{F}^{-1} \{ \hat{u}(\omega, \tau) \} = \mathcal{F}^{-1} \{ \hat{f}(\omega)e^{-\omega^2\tau} \} \]

\[ u(x, \tau) = (e^{(k+1)x/2} - e^{(k-1)x/2}) \ast \frac{1}{2\sqrt{\pi \tau}} e^{-x^2/(4\tau)} \]

\[ = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{\infty} \left( e^{(k+1)\frac{z}{2}} - e^{(k-1)\frac{z}{2}} \right) e^{-\frac{(x-z)^2}{4\tau}} \, dz \]
Undoing the Change of Variables (1 of 5)

Make the substitutions:

\[ z = x + \sqrt{2\tau}y \]
\[ dz = \sqrt{2\tau} \, dy \]

then

\[
\begin{align*}
    u(x, \tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \left( e^{(k+1)\frac{z}{2}} - e^{(k-1)\frac{z}{2}} \right) + e^{-(\frac{x-z)^2}{4\tau}} \, dz \\
    &= \frac{e^{(k+1)x/2}e^{(k+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k+1)\sqrt{2\tau})^2/2} \, dy \\
    &\quad - \frac{e^{(k-1)x/2}e^{(k-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k-1)\sqrt{2\tau})^2/2} \, dy
\end{align*}
\]
Now make the substitutions $w = y - \frac{1}{2}(k + 1)\sqrt{2\tau}$ in the first integral and $w' = y - \frac{1}{2}(k - 1)\sqrt{2\tau}$ in the second.

$u(x, \tau) = \frac{e^{(k+1)x/2}e^{(k+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k+1)\sqrt{2\tau})^2/2} \, dy$

$- \frac{e^{(k-1)x/2}e^{(k-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k-1)\sqrt{2\tau})^2/2} \, dy$

$= e^{(k+1)x/2+(k+1)^2\tau/4} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k + 1)\sqrt{2\tau} \right)$

$- e^{(k-1)x/2+(k-1)^2\tau/4} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k - 1)\sqrt{2\tau} \right)$

Recall: $\Phi$ is the cumulative normal distribution function.
Now make the substitutions \( w = y - \frac{1}{2}(k + 1)\sqrt{2\tau} \) in the first integral and \( w' = y - \frac{1}{2}(k - 1)\sqrt{2\tau} \) in the second.

\[
\begin{align*}
\frac{\text{u}(x, \tau)}{\sqrt{2\pi}} &= \frac{e^{(k+1)x/2}e^{(k+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k+1)\sqrt{2\tau})^2/2} \, dy \\
- \frac{e^{(k-1)x/2}e^{(k-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-\frac{1}{2}(k-1)\sqrt{2\tau})^2/2} \, dy \\
= e^{(k+1)x/2+(k+1)^2\tau/4} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k + 1)\sqrt{2\tau} \right) \\
- e^{(k-1)x/2+(k-1)^2\tau/4} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k - 1)\sqrt{2\tau} \right)
\end{align*}
\]

Recall: \( \Phi \) is the cumulative normal distribution function.
Note that
\[ e^{(k+1) \frac{x}{2} + (k+1)^2 \frac{\tau}{4}} e^{-(k-1) \frac{x}{2} - (k+1)^2 \frac{\tau}{4}} = e^x \]
\[ e^{(k-1) \frac{x}{2} + (k-1)^2 \frac{\tau}{4}} e^{-(k-1) \frac{x}{2} - (k+1)^2 \frac{\tau}{4}} = e^{k \tau} \]
and therefore
\[ v(x, \tau) = e^{-(k-1)x/2-(k+1)^2\tau/4} u(x, \tau) \]
\[ = e^x \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} \right) \]
\[ - e^{-k\tau} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau} \right). \]
Undoing the Change of Variables (4 of 5)

Recall that

\[ x = \ln \frac{S}{K} \]
\[ \tau = \frac{\sigma^2}{2}(T - t) \]
\[ k = \frac{2r}{\sigma^2} \]

and thus

\[ \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k + 1)\sqrt{2\tau} = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} = w \]
\[ \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k - 1)\sqrt{2\tau} = w - \sigma\sqrt{T - t}. \]
Undoing the Change of Variables (5 of 5)

\[ \nu(x, \tau) = \frac{S}{K} \Phi(w) - e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right) \]

\[ F(S, t) = S \Phi(w) - Ke^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right) \]

\[ w = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]
Undoing the Change of Variables (5 of 5)

\[ v(x, \tau) = \frac{S}{K} \Phi(w) - e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right) \]

\[ F(S, t) = S \Phi(w) - Ke^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right) \]

\[ w = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]

Finally we have the formula for the European call.

\[ C(S, t) = S \Phi(w) - K e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right) \]
Undoing the Change of Variables (5 of 5)

\[
\nu(x, \tau) = \frac{S}{K} \Phi(w) - e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right)
\]

\[
F(S, t) = S \Phi(w) - Ke^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right)
\]

\[
w = \ln \left( \frac{S}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T-t) \frac{1}{\sigma \sqrt{T-t}}
\]

Finally we have the formula for the European call.

\[
C(S, t) = S \Phi(w) - K e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right)
\]

Using the Put-Call Parity Formula we can find the formula for the European put.

\[
P(S, t) = K e^{-r(T-t)} \Phi \left( \sigma \sqrt{T-t} - w \right) - S \Phi(-w)
\]
Plotting the Call Price
Plotting the Put Price
Suppose the current price of a security is $62 per share. The continuously compounded interest rate is 10% per year. The volatility of the price of the security is $\sigma = 20\%$ per year. Find the cost of a five-month European call option with a strike price of $60 per share.
Example (2 of 2)

Summary:

\[ T = 5/12, \quad t = 0, \quad r = 0.10, \]
\[ \sigma = 0.20, \quad S = 62, \quad \text{and} \quad K = 60. \]
Example (2 of 2)

Summary:

\[ T = \frac{5}{12}, \quad t = 0, \quad r = 0.10, \]
\[ \sigma = 0.20, \quad S = 62, \quad \text{and} \quad K = 60. \]

\[ w = \ln \left( \frac{S}{K} \right) + \left( r + \sigma^2/2 \right) (T - t) \]
\[ \frac{\sigma \sqrt{T - t}}{\sigma \sqrt{T - t}} \]
\[ C = S \Phi (w) - Ke^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T - t} \right) \]
Example (2 of 2)

Summary:

\[ T = \frac{5}{12}, \quad t = 0, \quad r = 0.10, \]
\[ \sigma = 0.20, \quad S = 62, \quad \text{and} \quad K = 60. \]

\[ w = \ln\left(\frac{S}{K}\right) + \left( r + \frac{\sigma^2}{2} \right) (T - t) \frac{1}{\sigma \sqrt{T - t}} \approx 0.641287 \]
\[ C = S\Phi (w) - Ke^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T - t} \right) \approx 5.80 \]
Example

Suppose the current price of a security is $97 per share. The continuously compounded interest rate is 8% per year. The volatility of the price of the security is $\sigma = 45\%$ per year. Find the cost of a three-month European put option with a strike price of $95 per share.
Example

Summary:

\[ T = \frac{1}{4}, \quad t = 0, \quad r = 0.08, \]
\[ \sigma = 0.45, \quad S = 97, \quad \text{and} \quad K = 95. \]
Example

Summary:

\[ T = \frac{1}{4}, \quad t = 0, \quad r = 0.08, \]
\[ \sigma = 0.45, \quad S = 97, \quad \text{and} \quad K = 95. \]

\[
    w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}
\]

\[
    P = Ke^{-r(T-t)} \Phi \left( \sigma \sqrt{T - t - w} \right) - S \Phi (-w)
\]
Example

Summary:

\[ T = \frac{1}{4}, \quad t = 0, \quad r = 0.08, \]
\[ \sigma = 0.45, \quad S = 97, \quad \text{and} \quad K = 95. \]

\[ w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \approx 0.293985 \]
\[ P = Ke^{-r(T-t)}\Phi \left( \sigma \sqrt{T - t - w} \right) - S\Phi (-w) \approx 6.71 \]
Each financial firm writing option contracts may have its own estimate of the volatility \( \sigma \) of a stock. If we know the price of a call option, its strike price, expiry, the current stock price, and the risk-free interest rate, we can determine the implied volatility of the stock.
Implied Volatility (1 of 3)

Each financial firm writing option contracts may have its own estimate of the volatility $\sigma$ of a stock. If we know the price of a call option, its strike price, expiry, the current stock price, and the risk-free interest rate, we can determine the implied volatility of the stock.

Example
Suppose the current price of a security is $60 per share. The continuously compounded interest rate is 6.25% per year. The cost of a four-month European call option with a strike price of $62 per share is $3. What is the implied volatility of the stock?
We must solve the equation

\[
C = S \Phi(w) - Ke^{-rT} \Phi(w - \sigma\sqrt{T})
\]

\[
3 = 60\Phi\left(\frac{\left(0.0625 + \frac{\sigma^2}{2}\right) \frac{4}{12} + \ln \frac{60}{62}}{\sigma \sqrt{\frac{4}{12}}} \right)
- 62e^{-(0.0625) \frac{4}{12}} \Phi\left(\frac{\left(0.0625 + \frac{\sigma^2}{2}\right) \frac{4}{12} + \ln \frac{60}{62}}{\sigma \sqrt{\frac{4}{12}}} - \sigma \sqrt{\frac{4}{12}} \right).
\]
Using Newton’s Method, $\sigma \approx 0.241045$. 
The binomial model is a discrete approximation to the Black-Scholes initial value problem originally developed by Cox, Ross, and Rubinstein.

**Assumptions:**

- Strike price of the call option is $K$.
- Exercise time of the call option is $T$.
- Present price of the security is $S(0)$.
- Continuously compounded interest rate is $r$.
- Price of the security follows a geometric Brownian motion with variance $\sigma^2$.
- Present time is $t$. 
Binomial Lattice

If the value of the stock is $S(0)$ then at $t = T$

$$S(T) = \begin{cases} 
  uS(0) & \text{with probability } p, \\
  dS(0) & \text{with probability } 1 - p 
\end{cases}$$

where $0 < d < 1 < u$ and $0 < p < 1$. 

$S(T) = u \ S(0)$

$S(T) = d \ S(0)$
Continuous model:

\[ dS = \mu S \, dt + \sigma S \, dW(t) \]

\[ d(\ln S) = (\mu - \frac{1}{2} \sigma^2) \, dt + \sigma \, dW(t) \]

\[ \mathbb{E}[\ln S(t)] = \ln S(0) + (\mu - \frac{1}{2} \sigma^2) t \]

\[ \mathbb{V}(\ln S(t)) = \sigma^2 t \]
Making the Continuous and Discrete Models Agree (1 of 2)

Continuous model:

\[
\begin{align*}
    dS &= \mu S \, dt + \sigma S \, dW(t) \\
    d(\ln S) &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma \, dW(t) \\
    \mathbb{E}[\ln S(t)] &= \ln S(0) + \left(\mu - \frac{1}{2}\sigma^2\right)t \\
    \mathbb{V}(\ln S(t)) &= \sigma^2 t
\end{align*}
\]

In the absence of arbitrage \( \mu = r \), i.e. the return on the security should be the same as the return on an equivalent amount in savings.
Making the Continuous and Discrete Models Agree (2 of 2)

\[
\ln S(0) + (r - \frac{1}{2}\sigma^2)\Delta t = p \ln(uS(0)) + (1 - p) \ln(dS(0))
\]

\[
(r - \frac{1}{2}\sigma^2)\Delta t = p \ln u + (1 - p) \ln d
\]
Making the Continuous and Discrete Models Agree (2 of 2)

\[
\ln S(0) + (r - \frac{1}{2} \sigma^2) \Delta t = p \ln(u S(0)) + (1 - p) \ln(d S(0))
\]

\[
(r - \frac{1}{2} \sigma^2) \Delta t = p \ln u + (1 - p) \ln d
\]

The variance in the returns in the continuous and discrete models should also agree.

\[
\sigma^2 \Delta t = p[\ln(u S(0))]^2 + (1 - p)[\ln(d S(0))]^2
\]

\[
- (p \ln(u S(0)) + (1 - p) \ln(d S(0)))^2
\]

\[
= p(1 - p) (\ln u - \ln d)^2
\]
We would like to write $p$, $u$, and $d$ as functions of $r$, $\sigma$, and $\Delta t$.

\[
\begin{align*}
p \ln u + (1 - p) \ln d &= (r - \frac{1}{2} \sigma^2) \Delta t \\
p(1 - p) (\ln u - \ln d)^2 &= \sigma^2 \Delta t
\end{align*}
\]

We need a third equation in order to solve this system. We are free to pick any equation consistent with the first two. We pick $d = 1/u$ (why?).
We would like to write $p$, $u$, and $d$ as functions of $r$, $\sigma$, and $\Delta t$.

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\[
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\]

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p(1 - p) (\ln u - \ln d)^2 = \sigma^2 \Delta t
\]

- We need a third equation in order to solve this system.
- We are free to pick any equation consistent with the first two.
- We pick $d = 1/u$ (why?).
Solving the System

\[(2p - 1) \ln u = (r - \frac{1}{2} \sigma^2) \Delta t\]

\[4p(1 - p)(\ln u)^2 = \sigma^2 \Delta t\]

1. Square the first equation and add to the second.
2. Ignore terms involving \((\Delta t)^2\).
Solving the System

\[(2p - 1) \ln u = (r - \frac{1}{2} \sigma^2) \Delta t\]
\[4p(1 - p)(\ln u)^2 = \sigma^2 \Delta t\]

1. Square the first equation and add to the second.
2. Ignore terms involving \((\Delta t)^2\).

\[u = e^{\sigma \sqrt{\Delta t}}\]
\[d = e^{-\sigma \sqrt{\Delta t}}\]
\[p = \frac{1}{2} \left(1 + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) \sqrt{\Delta t}\right)\]
Example

Suppose $S(0) = 1$, $r = 0.10$, $\sigma = 0.20$, $T = 1/4$, $\Delta t = 1/12$, then the lattice of security prices resembles:
Determining a European Call Price

Payoff: $(S(T) - K)^+$

Let $Y$ be a binomial random variable with probability of an UP step $p$ and $n$ total steps.

\[
C = e^{-rT} \mathbb{E} \left[ (u^Y d^{n-Y} S(0) - K)^+ \right]
\]

\[
= e^{-rT} \mathbb{E} \left[ (e^{Y\sigma\sqrt{\Delta t}} e^{(Y-n)\sigma\sqrt{\Delta t}} S(0) - K)^+ \right]
\]

\[
= e^{-rT} \mathbb{E} \left[ (e^{(2Y-n)\sigma\sqrt{\Delta t}} S(0) - K)^+ \right]
\]

\[
= e^{-rT} \mathbb{E} \left[ (e^{(2Y-T/\Delta t)\sigma\sqrt{\Delta t}} S(0) - K)^+ \right].
\]
The price of a security is $62, the continuously compounded interest rate is 10% per year, the volatility of the price of the security is $\sigma = 20\%$ per year. If the strike price of a call option is $60$ per share with an expiry of 5 months, then $C = 5.789$ according to the solution to the Black-Scholes equation.

The parameters of the discrete model are:

$$u = 1.05943, \quad d = 0.9439, \quad \text{and} \quad p = 0.557735.$$
Lattice of Security Prices
Payoffs of the Call Option

\[ S | (S - K)^+ \] | Prob. \\
--- | --- | --- \\
82.7488 | 22.7488 | \( \binom{5}{5} u^5 d^0 \approx 0.0539684 \) \\
73.7248 | 13.7248 | \( \binom{4}{4} u^4 d^1 \approx 0.213976 \) \\
65.6849 | 5.6849 | \( \binom{3}{3} u^3 d^2 \approx 0.339351 \) \\
58.5218 | 0 | \( \binom{2}{2} u^2 d^3 \approx 0.269094 \) \\
52.1398 | 0 | \( \binom{1}{1} u^1 d^3 \approx 0.106691 \) \\
46.4538 | 0 | \( \binom{0}{0} u^0 d^5 \approx 0.0169205 \)

\[
C \approx \frac{(5.6849)(0.3394) + (13.7248)(0.2140) + (22.7488)(0.0540)}{e^{(0.10)(5/12)}} \\
= 5.83509.
\]
These slides are adapted from the textbook, 


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