The Black-Scholes Equation

MATH 472 *Financial Mathematics*

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Objectives

In this lesson we will:

- derive the Black-Scholes partial differential equation using Itô's Lemma and no-arbitrage assumptions,
- determine the appropriate boundary conditions for European options,
- summarize the initial, boundary value problem for European options.

Note: in this presentation the notation $dW(t)$ will be used in place of dW_t since this derivation will make extensive use of partial derivatives.

Assumptions

- Price of the security follows the log-normal random walk.
- Risk-free interest rate *r* and volatility of security σ are known functions of time.
- No transaction costs.
- Security pays no dividends.
- No arbitrage, trading of assets takes place continuously, short selling is possible, and fractions of an asset can be sold.

Black-Scholes Equation

We have already approximated the price of a European call option using a multi-step binomial model. Now we will use the continuous stochastic model of stock prices.

- Suppose a stock obeys an Itô process of the form:

$$dS = \mu S\,dt + \sigma S\,dW(t)$$

- An investor will create a portfolio Y, consisting of a short position in a European call option and a long position of Δ shares of the stock.

$$Y = F(S,t) = C^e(S,t) - (\Delta)S$$

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- Use Itô's lemma to find the stochastic process followed by Y.

Recall Itô's Lemma

Lemma (Itô's Lemma)

Suppose that the random variable X is described by the Itô process

$$dS = a(S,t)\,dt + b(S,t)\,dW(t)$$

where $dW(t)$ is the differential of the standard Wiener process. Suppose the random variable $Y = F(S,t)$. Then Y is described by the following Itô process.

$$dY = \left(a(S,t)F_S + F_t + \frac{1}{2}(b(S,t))^2 F_{SS} \right) dt + b(S,t)F_S dW(t)$$

Partial Derivatives

$$\begin{aligned} F(S,t) &= C^e(S,t) - (\Delta)S \\ F_S &= C^e_S - \Delta \\ F_{SS} &= C^e_{SS} \\ F_t &= C^e_t \end{aligned}$$

Itô Process for dY

Let $a(S,t) = \mu S$ and $b(S,t) = \sigma S$ then

$$
\begin{aligned}
dY &= \left(a(S,t) F_S + F_t + \frac{1}{2}(b(S,t))^2 F_{SS} \right) dt + b(S,t) F_S dW(t) \\
&= \left(\mu S \left(C^e_S - \Delta \right) + \frac{1}{2}\sigma^2 S^2 C^e_{SS} + C^e_t \right) dt + \sigma S \left(C^e_S - \Delta \right) dW(t)
\end{aligned}
$$

Itô Process for dY

Let $a(S,t) = \mu\, S$ and $b(S,t) = \sigma\, S$ then

$$\begin{aligned} dY &= \left(a(S,t)F_S + F_t + \frac{1}{2}(b(S,t))^2 F_{SS} \right) dt + b(S,t)F_S dW(t) \\ &= \left(\mu S\left(C_S^e - \Delta\right) + \frac{1}{2}\sigma^2 S^2 C_{SS}^e + C_t^e \right) dt + \sigma S\left(C_S^e - \Delta\right) dW(t) \end{aligned}$$

Note: the process becomes deterministic if $\Delta = C_S^e$.

$$dY = \left(\frac{1}{2}\sigma^2 S^2 C_{SS}^e + C_t^e \right) dt$$

No Arbitrage Assumption

The payoff from the portfolio should be the same as that generated by investing an amount of cash equal to Y in savings earning interest compounded continuously at rate r.

$$\begin{aligned} Y &= Y_0 e^{r\,t} \\ dY &= r\,Y_0 e^{r\,t}\,dt = r\,Y\,dt \\ &= r(C^e - (\Delta)S)\,dt \\ &= r\left(C^e - C^e_S S\right)\,dt \end{aligned}$$

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Recall from Itô's lemma that

$$dY = \left(\frac{1}{2}\sigma^2 S^2 C^e_{SS} + C^e_t\right)\,dt.$$

Equating the two expressions for dY yields the **Black-Scholes partial differential equation**

$$r\,C^e = C^e_t + r\,S\,C^e_S + \frac{1}{2}\sigma^2 S^2 C^e_{SS}.$$

Final and Boundary Conditions (1 of 2)

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this is the final condition.

The stock will have a value in the interval $[0, \infty)$. The boundary at $S = 0$ is absorbing, so if there is a time $t^* \geq 0$ such that $S(t^*) = 0$, then $S(t) = 0$ for all $t \geq t^*$. In this case the option will never be exercised and is worthless. Thus

$$C^e(0, t) = 0,$$

which is the boundary condition at $S = 0$.

Boundary Conditions (2 of 2)

From the Put-Call Parity Formula:

$$\begin{aligned} C^e &= P^e + S - Ke^{-rT} \\ \lim_{S\to\infty} C^e &= \lim_{S\to\infty} P^e + S - Ke^{-rT} \\ C^e &\to S - Ke^{-rT} \quad \text{as } S \to \infty. \end{aligned}$$

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As the security grows unbounded in value:

- a put option (right to sell at a finite price) becomes worthless, and
- the call option is worth the difference between the security price and the present value of the strike.

IBVP: European Call

For (S, t) in $[0, \infty) \times [0, T]$,

$$
\begin{aligned}
r\,C^e &= C^e_t + r\,S\,C^e_S + \frac{1}{2}\sigma^2 S^2 C^e_{SS} \\
C^e(S, T) &= (S(T) - K)^+ \text{ for } S > 0, \\
C^e(0, t) &= 0 \text{ for } 0 \le t < T, \\
C^e(S, t) &= S - K\,e^{-r(T-t)} \text{ as } S \to \infty.
\end{aligned}
$$

European Put

If an investor creates a portfolio by buying a European put and shorting Δ shares of the security, then the steps followed above for the European call produce the PDE:

$$r\,P^e = P^e_t + r\,S\,P^e_S + \frac{1}{2}\sigma^2 S^2 P^e_{SS}.$$

Question: What are the appropriate final and boundary conditions for a European put?

IBVP: European Put

For (S, t) in $[0, \infty) \times [0, T]$,

$$\begin{aligned} r\,P^e &= P^e_t + r\,S\,P^e_S + \frac{1}{2}\sigma^2 S^2 P^e_{SS} \\ P^e(S,T) &= (K - S(T))^+ \text{ for } S > 0, \\ P^e(0,t) &= K\,e^{-r(T-t)} \text{ for } 0 \le t < T, \\ P^e(S,t) &= 0 \text{ as } S \to \infty. \end{aligned}$$

Effect of Continuous Dividends

Assumption: the stock pays dividends at a continuous rate proportional to the value of the stock

$$\text{dividend per unit time} = \delta S$$

How much dividend is paid in a short time interval dt?

$$\text{dividend paid} = \delta S \, dt$$

What stochastic differential equation would the value of the stock paying a continuous proportional dividend obey?

$$dS = (\mu - \delta) S \, dt + \sigma S \, dW(t)$$

Arbitrage-free Portfolio

Suppose $C^{e,\delta}(S,t)$ is the value of a European call option on the stock paying a continuous dividend.

As before, create a portfolio of a short position in the call option and a long position in Δ shares of the stock.

$$Y = C^{e,\delta} - (\Delta)S$$

Change in Portfolio Value

One share of stock pays $\delta S\,dt$ in dividends during a time interval of length dt, thus Δ shares of stock pays

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Using Itô's Lemma, the portfolio changes in value

$$\begin{aligned} dY &= d(C^{e,\delta} - (\Delta)S) - \delta(\Delta)S\, dt \\ &= \left((\mu - \delta)SC^{e,\delta}_S + \frac{1}{2}\sigma^2 S^2 C^{e,\delta}_{SS} + C^{e,\delta}_t \right) dt + \sigma S C^{e,\delta}_S\, dW(t) \\ &\quad - (\Delta)\left((\mu - \delta)S\, dt + \sigma S\, dW(t)\right) - \delta(\Delta)S\, dt \\ &= \left((\mu - \delta)S(C^{e,\delta}_S - \Delta) + \frac{1}{2}\sigma^2 S^2 C^{e,\delta}_{SS} + C^{e,\delta}_t - \delta(\Delta)S \right) dt \\ &\quad + \sigma S(C^{e,\delta}_S - \Delta)\, dW(t). \end{aligned}$$

Eliminating Randomness

Choose $\Delta = C_S^{e,\delta}$ and the portfolio obeys the stochastic differential equation:

$$dY = \left(\frac{1}{2}\sigma^2 S^2 C_{SS}^{e,\delta} + C_t^{e,\delta} - \delta\, S\, C_S^{e,\delta} \right) dt.$$

In the absence of arbitrage the change in the value of the portfolio should be the same as the interest earned by a equivalent amount of cash.

$$dY = r(C^{e,\delta} - (\Delta)S)\, dt$$

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Thus the Black-Scholes partial differential equation for the stock paying continuous dividends becomes

$$r\, C^{e,\delta} = C_t^{e,\delta} + \frac{1}{2}\sigma^2 S^2 C_{SS}^{e,\delta} + (r - \delta) S\, C_S^{e,\delta}.$$

Similarities with Non-Dividend-Paying Stocks

- Payoff of the call option at expiry:
  $C^{e,\delta}(S,T) = (S(T) - K)^+$.
- Boundary condition at $S = 0$ is $C^{e,\delta}(0,t) = 0$.
- Boundary condition as $S \to \infty$:

$$
\begin{aligned}
C^{e,\delta}(S,t) &= P^e + S\, e^{-\delta(T-t)} - K\, e^{-r(T-t)} \\
\lim_{S\to\infty} C^{e,\delta}(S,t) &= \lim_{S\to\infty} \left(P^e + S\, e^{-\delta(T-t)} - K\, e^{-r(T-t)} \right) \\
&= S\, e^{-\delta(T-t)} - K\, e^{-r(T-t)}.
\end{aligned}
$$

IBVP: European Call with Continuous Dividends

For (S, t) in $[0, \infty) \times [0, T]$,

$$\begin{aligned} r\,C^{e,\delta} &= C_t^{e,\delta} + (r-\delta)\,S\,C_S^{e,\delta} + \frac{1}{2}\sigma^2 S^2 C_{SS}^{e,\delta} \\ C^{e,\delta}(S,T) &= (S(T)-K)^+ \text{ for } S > 0, \\ C^{e,\delta}(0,t) &= 0 \text{ for } 0 \le t < T, \\ C^{e,\delta}(S,t) &= S\,e^{-\delta(T-t)} - K\,e^{-r(T-t)} \text{ as } S \to \infty. \end{aligned}$$

Homework

- Read Sections 7.4, 7.5
- Exercises: 10

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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