Solution of the Black-Scholes Equation

MATH 472 *Financial Mathematics*

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Objectives

In this lesson we will learn to:

- change variables in the Black-Scholes partial differential equation in order to transform it into a form more easily solvable,
- solve the unbounded heat equation,
- price the European call and put options on non-dividend-paying securities using the Black-Scholes formula.

Initial Boundary Value Problem for the Heat Equation

The final form of the Black-Scholes IBVP for a European call option on a non-dividend-paying stock is as follows.

$$
\begin{aligned}
u_\tau &= u_{xx} \text{ for } x \in (-\infty, \infty) \text{ and } \tau \in (0, T\sigma^2/2) \\
u(x,0) &= \left(e^{(k+1)x/2} - e^{(k-1)x/2}\right)^+ \text{ for } x \in (-\infty, \infty) \\
u(x,\tau) &\to 0 \text{ as } x \to -\infty \text{ for } \tau \in (0, T\sigma^2/2) \\
u(x,\tau) &\to e^{\frac{(k+1)}{2}[x+(k+1)\tau/2]} - e^{\frac{(k-1)}{2}[x+(k-1)\tau/2]} \\
&\quad \text{as } x \to \infty \text{ for } \tau \in (0, T\sigma^2/2)
\end{aligned}
$$

Fundamental Solution to the Heat Equation

Define the **fundamental solution to the heat equation** as

$$U(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

Solving the Black-Scholes IVP

Theorem

Consider the initial value problem

$$\begin{aligned} u_t &= u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x,0) &= f(x), \text{ for } -\infty < x < \infty. \end{aligned}$$

If $f(x)$ is continuous and if $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, then the piecewise defined function

$$u(x,t) = \begin{cases} \int_{-\infty}^{\infty} U(x-y,t) f(y)\, dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0 \end{cases}$$

solves the heat equation and satisfies the initial condition in the sense that

$$\lim_{(x,t)\to(x_0,0^+)} u(x,t) = f(x_0).$$

Superposition of Solutions

$$
\begin{aligned}
u(x,\tau) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-(x-y)^2/(4\tau)} \left(e^{(k+1)y/2} - e^{(k-1)y/2} \right)^{+} \, dy \\
&= \frac{1}{\sqrt{4\pi\tau}} \int_{0}^{\infty} e^{-(x-y)^2/(4\tau)} \left(e^{(k+1)y/2} - e^{(k-1)y/2} \right) \, dy \\
&= \frac{1}{\sqrt{4\pi\tau}} e^{-x^2/(4\tau)} \int_{0}^{\infty} e^{-(y^2-2(x+(k+1)\tau)y)/(4\tau)} \, dy \\
&\quad - \frac{1}{\sqrt{4\pi\tau}} e^{-x^2/(4\tau)} \int_{0}^{\infty} e^{-(y^2-2(x+(k-1)\tau)y)/(4\tau)} \, dy
\end{aligned}
$$

Complete the square in y in each exponent.

$$
\begin{aligned}
u(x,\tau) &= \frac{1}{\sqrt{4\pi\tau}} e^{-(x^2-(x+(k+1)\tau)^2)/(4\tau)} \int_0^\infty e^{-(y-x-(k+1)\tau)^2/(4\tau)}\,dy \\
&\quad - \frac{1}{\sqrt{4\pi\tau}} e^{-(x^2-(x+(k-1)\tau)^2)/(4\tau)} \int_0^\infty e^{-(y-x-(k-1)\tau)^2/(4\tau)}\,dy \\
&= \frac{1}{\sqrt{4\pi\tau}} e^{(2x(k+1)+(k+1)^2\tau)/4} \int_0^\infty e^{-(y-x-(k+1)\tau)^2/(4\tau)}\,dy \\
&\quad - \frac{1}{\sqrt{4\pi\tau}} e^{(2x(k-1)+(k-1)^2\tau)/4} \int_0^\infty e^{-(y-x-(k-1)\tau)^2/(4\tau)}\,dy
\end{aligned}
$$

Make the substitutions $s = \dfrac{y-x-(k+1)\tau}{\sqrt{2\tau}}$ in the first integral and $z = \dfrac{y-x-(k-1)\tau}{\sqrt{2\tau}}$ in the second integral.

$$
\begin{aligned}
u(x,\tau) &= e^{(2x(k+1)+(k+1)^2\tau)/4}\frac{1}{\sqrt{2\pi}}\int_{-(x+(k+1)\tau)/\sqrt{2\tau}}^{\infty} e^{-s^2/2}\,ds \\
&\quad - e^{(2x(k-1)+(k-1)^2\tau)/4}\frac{1}{\sqrt{2\pi}}\int_{-(x+(k-1)\tau)/\sqrt{2\tau}}^{\infty} e^{-z^2/2}\,dz \\
&= e^{(2x(k+1)+(k+1)^2\tau)/4}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(x+(k+1)\tau)/\sqrt{2\tau}} e^{-s^2/2}\,ds \\
&\quad - e^{(2x(k-1)+(k-1)^2\tau)/4}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(x+(k-1)\tau)/\sqrt{2\tau}} e^{-z^2/2}\,dz
\end{aligned}
$$

Cumulative Distribution Function

Define the function $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz$, then

$$\begin{aligned} u(x,\tau) &= e^{(2x(k+1)+(k+1)^2\tau)/4} \Phi\left(\frac{x+(k+1)\tau}{\sqrt{2\tau}}\right) \\ &\quad - e^{(2x(k-1)+(k-1)^2\tau)/4} \Phi\left(\frac{x+(k-1)\tau}{\sqrt{2\tau}}\right) \end{aligned}$$

Recall that $v(x,\tau) = e^{-(k-1)x/2-(k+1)^2\tau/4} u(x,\tau)$ and find $v(x,\tau)$.

Reversing the Change of Variables (1 of 2)

$$v(x,\tau) = e^x \Phi\left(\frac{x+(k+1)\tau}{\sqrt{2\tau}}\right) - e^{-k\tau}\Phi\left(\frac{x+(k-1)\tau}{\sqrt{2\tau}}\right)$$

Recall that

$$\begin{aligned}
k &= \frac{2r}{\sigma^2} \\
x &= \ln\frac{S}{K} \\
\tau &= \frac{\sigma^2}{2}(T-t) \\
v(x,\tau) &= \frac{F(S,t)}{K}
\end{aligned}$$

and find $F(S,t)$.

Reversing the Change of Variables (2 of 2)

$$
\begin{aligned}
v(x,\tau) &= e^{x}\Phi\left(\frac{x+(k+1)\tau}{\sqrt{2\tau}}\right) - e^{-k\tau}\Phi\left(\frac{x+(k-1)\tau}{\sqrt{2\tau}}\right) \\
\frac{F(S,t)}{K} &= \frac{S}{K}\Phi\left(\frac{\ln\frac{S}{K}+\left(\frac{2r}{\sigma^2}+1\right)\frac{\sigma^2}{2}(T-t)}{\sqrt{\sigma^2(T-t)}}\right) \\
&\quad - e^{-\frac{2r}{\sigma^2}\frac{\sigma^2}{2}(T-t)}\Phi\left(\frac{\ln\frac{S}{K}+\left(\frac{2r}{\sigma^2}-1\right)\frac{\sigma^2}{2}(T-t)}{\sqrt{\sigma^2(T-t)}}\right) \\
F(S,t) &= S\,\Phi\left(\frac{\ln\frac{S}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - K\,e^{-r(T-t)}\Phi\left(\frac{\ln\frac{S}{K}+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

Black-Scholes Formula for a European Call

If S is a non-dividend-paying security with volatility σ and the risk-free interest rate is r, a K-strike European call option with expiry at T has price:

$$\begin{aligned} C^e(S,t) &= S\,\Phi\,(w) - K\,e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\ \text{where } w &= \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

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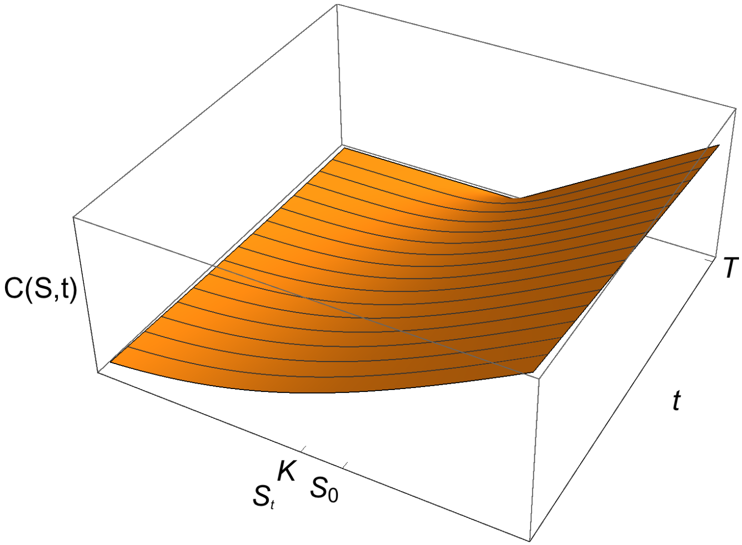
\text{where } w &= \frac{\ln \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}
$$

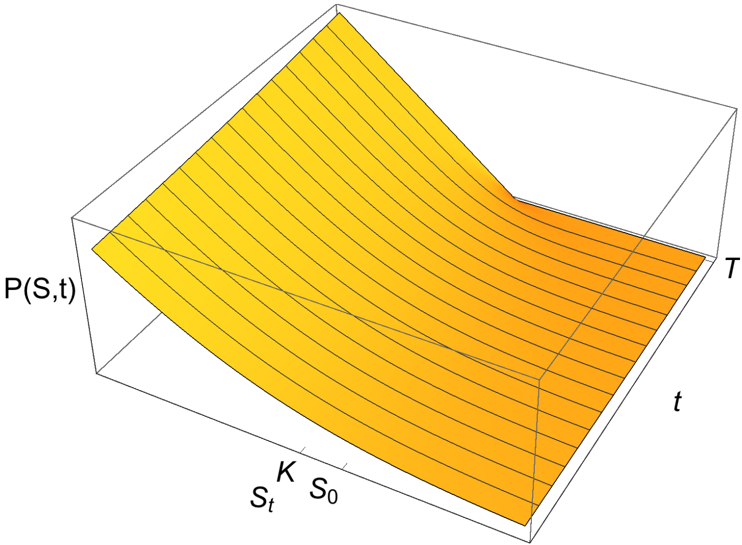
Using the Put-Call Parity Formula we can find the formula for the European put.

$$
P^e(S,t) = K\,e^{-r(T-t)}\Phi\left(\sigma\sqrt{T-t} - w\right) - S\,\Phi\left(-w\right)
$$

Plotting the Call Price



Plotting the Put Price



Example (1 of 2)

Suppose the current price of a security is \$62 per share. The continuously compounded interest rate is 10% per year. The volatility of the price of the security is $\sigma = 20\%$ per year. Find the cost of a five-month European call option with a strike price of \$60 per share.

Example (2 of 2)

Summary:

$$
\begin{aligned}
T &= 5/12, \quad t = 0, \quad r = 0.10, \\
\sigma &= 0.20, \quad S = 62, \quad \text{and} \quad K = 60.
\end{aligned}
$$

Example (2 of 2)

Summary:

$$\begin{aligned} T &= 5/12, \quad t = 0, \quad r = 0.10, \\ \sigma &= 0.20, \quad S = 62, \quad \text{and} \quad K = 60. \end{aligned}$$

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\ C^e &= S\Phi\,(w) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T - t}\right) \end{aligned}$$

Example (2 of 2)

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\sigma &= 0.20, \quad S = 62, \quad \text{and} \quad K = 60.
\end{aligned}
$$

$$
\begin{aligned}
w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \approx 0.641287 \\
C^e &= S\Phi(w) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T - t}\right) \approx \$5.80
\end{aligned}
$$

Example

Suppose the current price of a security is \$97 per share. The continuously compounded interest rate is 8% per year. The volatility of the price of the security is $\sigma = 45\%$ per year. Find the cost of a three-month European put option with a strike price of \$95 per share.

Example

Summary:

$$\begin{aligned} T &= 1/4, \quad t = 0, \quad r = 0.08, \\ \sigma &= 0.45, \quad S = 97, \quad \text{and} \quad K = 95. \end{aligned}$$

Example

Summary:

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\begin{aligned}
T &= 1/4, \quad t = 0, \quad r = 0.08, \\
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$$

$$
\begin{aligned}
w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\
P^e &= Ke^{-r(T-t)}\Phi\left(\sigma\sqrt{T - t} - w\right) - S\Phi\left(-w\right)
\end{aligned}
$$

Example

Summary:

$$\begin{aligned} T &= 1/4, \quad t = 0, \quad r = 0.08, \\ \sigma &= 0.45, \quad S = 97, \quad \text{and} \quad K = 95. \end{aligned}$$

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \approx 0.293985 \\ P^e &= Ke^{-r(T-t)}\Phi\left(\sigma\sqrt{T - t} - w\right) - S\Phi\left(-w\right) \approx \$6.71 \end{aligned}$$

Implied Volatility (1 of 3)

Each financial firm writing option contracts may have its own estimate of the volatility σ of a stock. If we know the price of a call option, its strike price, expiry, the current stock price, and the risk-free interest rate, we can determine the **implied volatility** of the stock.

Implied Volatility (1 of 3)

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Example

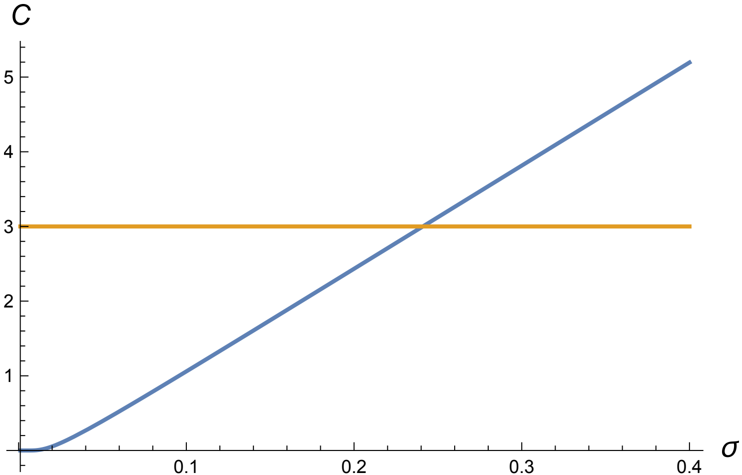
Suppose the current price of a security is \$60 per share. The continuously compounded interest rate is 6.25% per year. The cost of a four-month European call option with a strike price of \$62 per share is \$3. What is the implied volatility of the stock?

Implied Volatility (2 of 3)

We must solve the equation

$$
\begin{aligned}
C^e &= S\Phi(w) - Ke^{-rT}\Phi\left(w - \sigma\sqrt{T}\right) \\
3 &= 60\Phi\left(\frac{\left(0.0625 + \frac{\sigma^2}{2}\right)\frac{4}{12} + \ln\frac{60}{62}}{\sigma\sqrt{\frac{4}{12}}}\right) \\
&\quad - 62e^{-(0.0625)\frac{4}{12}}\Phi\left(\frac{\left(0.0625 + \frac{\sigma^2}{2}\right)\frac{4}{12} + \ln\frac{60}{62}}{\sigma\sqrt{\frac{4}{12}}} - \sigma\sqrt{\frac{4}{12}}\right).
\end{aligned}
$$

Implied Volatility (3 of 3)



Using Newton's Method, $\sigma \approx 0.241045$.

Homework

- Read Section 8.4 (omit discussion of the Fourier Transform)
- Exercises: 17–19

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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