Continuous Random Variables

MATH 472 *Financial Mathematics*

J Robert Buchanan

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Objectives

In this lesson we will learn:

- the fundamental difference in interpretation of continuous random variables *versus* discrete random variables,
- how to assign probabilities to continuous random variables,
- properties of jointly distributed continuous random variables.

Discrete *vs.* Continuous Random Variables

Think about the probability of selecting X from the interval $[0, 1]$ when

- $X \in \{0, 1\}$
- $X \in \{k/10 : k = 0, 1, \ldots, 10\}$
- $X \in \{k/n : k = 0, 1, \ldots, n\}$ and $n \in \mathbb{N}$

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Question: what happens to the last probability as $n \to \infty$?

Continuous Random Variables

Definition

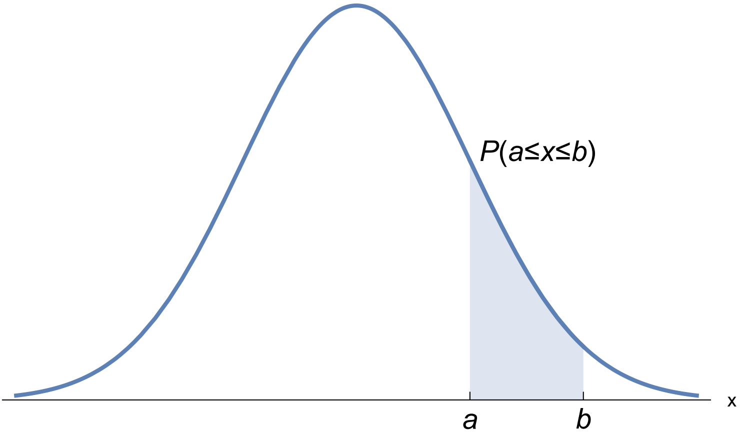
A random variable X has a **continuous distribution** (or **probability distribution function** or **probability density function**) if there exists a non-negative function $f_X : \mathbb{R} \to \mathbb{R}$ such that for an interval $[a, b]$ the

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\, dx.$$

The function f_X must have the following property,

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1.$$

Area Under the PDF



Remark: the area under the curve may be interpreted as probability.

Example

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Solution

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \frac{C}{1+x^2}\,dx \\
&= C\left[\tan^{-1} x\right]_{x\to-\infty}^{x\to\infty} \\
&= C\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\
C &= \frac{1}{\pi}
\end{aligned}$$

Uniformly Distributed Continuous Random Variables

Definition

A continuous random variable X is **uniformly distributed** in the interval $[a, b]$ (with $b > a$) if the probability that X belongs to any subinterval of $[a, b]$ is equal to the length of the subinterval divided by $b - a$.

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Question: Assuming the PDF vanishes outside of $[a, b]$ and is constant on $[a, b]$, what is the PDF?

Answer: $f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$

Example (1 of 2)

Random variable X is continuously and uniformly randomly distributed in the interval $[-5, 5]$. Find the probability that $-1 \leq X \leq 2$.

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$$\mathbb{P}(-1 \leq X \leq 2) = \frac{2-(-1)}{5-(-5)} = \frac{3}{10}$$

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$$\begin{aligned} \mathbb{P}\left((-3 \leq X \leq 1) \cup (X > 7)\right) &= \mathbb{P}(-3 \leq X \leq 1) + \mathbb{P}(X > 7) \\ &= \frac{1-(-3)}{10-(-10)} + \frac{10-7}{10-(-10)} \\ &= \frac{7}{20} \end{aligned}$$

Exponential Distribution

Definition

Let $\lambda > 0$, then random variable X has an **exponential distribution** provided its probability distribution function is

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Solution

$$\mathbb{P}\,(1 < X < 2) = \int_1^2 \lambda e^{-\lambda x}\, dx = \left[-e^{-\lambda x}\right]_{x=1}^{x=2} = e^{-\lambda} - e^{-2\lambda}$$

Cumulative Distribution Function

Definition

If X is a continuous random variable with probability distribution function $f_X(x)$, the **cumulative distribution function** (**CDF**) of X is

$$F(x) = \int_{-\infty}^{x} f_X(u)\, du.$$

If f_X is continuous at $u = x$, then $F'(x) = f_X(x)$.

Examples

Find the CDFs for the following random variables with the given probability distributions.

1. $f_X(x) = \lambda e^{-\lambda x}$ with $\lambda > 0$.

2. $f_Y(y) = \frac{1}{\pi(1+x^2)}$.

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$$F(y) = \int_{-\infty}^{y} \frac{1}{\pi(1+u^2)}\, du = \left[\frac{1}{\pi} tan^{-1} u\right]_{u \to -\infty}^{u=y} = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} y$$

Joint and Marginal Distributions

Definition

A **joint probability density** for a pair of random variables, X and Y, is a non-negative function $f_{X,Y}(x, y)$ for which

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = 1.$$

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If X and Y are continuous random variables with joint distribution $f_{X,Y}(x,y)$ then the **marginal density** for X is defined as the function

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Remark: a similar definition may be stated for the marginal density for Y.

Example

If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 1/\pi & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

find the marginal probability density of X.

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$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1, \\ 0 & \text{otherwise} \end{cases}$$

Independence of Jointly Distributed RVs

Definition

Two continuous random variables are **independent** if and only if the joint probability density function factors into the product of the marginal densities of X and Y. In other words X and Y are independent if and only if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

for all real numbers x and y.

Example

The joint probability density function of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} x\,y/2 & \text{if } 0 \le x \le y \text{ and } 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent?

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Are X and Y independent?

No, since $f_X(x) = x - x^3/4$ if $0 \le x \le 2$ and $f_Y(y) = y^3/4$ if $0 \le y \le 2$.

Example

Suppose X and Y are identically distributed independent random variables with common CDF F and PDF f. Let $M = \max\{X, Y\}$.

1. Find the CDF of random variable M.

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$$M \leq m \iff (X \leq m) \text{ and } (Y \leq m)$$

   This implies

$$\begin{aligned} \mathbb{P}(M \leq m) &= \mathbb{P}((X \leq m) \cap (Y \leq m)) \\ &= \mathbb{P}(X \leq m)\,\mathbb{P}(Y \leq m) \\ &= F(m)\,F(m) = (F(m))^2 \end{aligned}$$

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2. Find the PDF of random variable M.

$$f_M(m) = \frac{d}{dm}\left[(F(m))^2\right] = 2F(m)F'(m) = 2F(m)f(m)$$

Homework

- Read Sections 3.1 and 3.2
- Exercises: 1–4

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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