Expected Value and Variance of Continuous Random Variables

MATH 472 *Financial Mathematics*

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Objectives

In this lesson we will learn to:

- compute the expected value of a continuously distributed random variable,
- compute the variance of a continuously distributed random variable, and
- use the properties of expected value and variance of continuously distributed random variables.

Expected Value

Definition

The **expected value** or **mean** of a continuous random variable X with probability density function $f(x)$ is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

Example

Find the expected value of X, if X is a continuously uniformly distributed random variable on the interval $[-10, 80]$.

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$$\begin{aligned}\mathbb{E}(X) &= \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-10}^{80} \frac{x}{90}\, dx \\ &= \left.\frac{x^2}{180}\right|_{-10}^{80} = \frac{6400}{180} - \frac{100}{180} = 35\end{aligned}$$

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\end{aligned}
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Question: if X is a uniformly distributed but integer-valued RV, what is its expected value?

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Question: if X is a uniformly distributed but integer-valued RV, what is its expected value?

$$\mathbb{E}(X) = \sum_{x=-10}^{80} \frac{x}{91} = 35 \text{ (coincidentally)}$$

Example

If $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$, find $\mathbb{E}(X)$.

Example

If $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$, find $\mathbb{E}(X)$.

Solution

$$
\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^{\infty} \lambda x\, e^{-\lambda x}\, dx \\
&= \left[-xe^{-\lambda x}\right]_{x=0}^{x\to\infty} + \int_0^{\infty} e^{-\lambda x}\, dx \\
&= \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_{x=0}^{x\to\infty} = \frac{1}{\lambda}
\end{aligned}
$$

Example

If $f_X(x) = \frac{1}{\pi(1+x^2)}$, find $\mathbb{E}(X)$.

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Solution

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)}\, dx \\
&= \int_{-\infty}^{0} \frac{x}{\pi(1+x^2)}\, dx + \int_{0}^{\infty} \frac{x}{\pi(1+x^2)}\, dx \\
&= \int_{\infty}^{1} \frac{1}{2\pi u}\, du + \int_{1}^{\infty} \frac{1}{2\pi u}\, du
\end{aligned}$$

These integrals diverge, thus the Cauchy distribution does not possess an expected value.

Expected Value of a Function

Definition

The expected value of a function g of a continuously distributed random variable X which has probability density function f is defined as

$$\mathbb{E}\,(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\,dx,$$

provided the improper integral converges absolutely, *i.e.*, $\mathbb{E}\,(g(X))$ is defined if and only if

$$\int_{-\infty}^{\infty} |g(x)|f(x)\,dx < \infty.$$

Example

Find the expected value of X^2 if X is continuously distributed on $[0, \infty)$ with probability density function $f(x) = e^{-x}$.

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$$
\begin{aligned}
\mathbb{E}\left(X^2\right) &= \int_0^\infty x^2 e^{-x}\,dx \\
&= \lim_{M\to\infty} \int_0^M x^2 e^{-x}\,dx \\
&= \lim_{M\to\infty} \left[-(x^2+2x+2)e^{-x}\right]\Big|_0^M \\
&= \lim_{M\to\infty} \left[2-(M^2+2M+2)e^{-M}\right] \\
&= 2
\end{aligned}
$$

Example

An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss X, follows a distribution with probability density,

$$f_X(x) = \begin{cases} 2/x^3 & \text{if } x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the benefit paid under the insurance policy?

Solution

Let Y denote the benefit paid.

$$Y = \begin{cases} X & \text{if } 1 < X \leq 10 \\ 10 & \text{if } X > 10 \end{cases}$$

$$\begin{aligned} \mathbb{E}(Y) &= \int_1^{10} x \, \frac{2}{x^3} \, dx + \int_{10}^{\infty} 10 \, \frac{2}{x^3} \, dx \\ &= \left[-\frac{2}{x} \right]_{x=1}^{x=10} + \left[-\frac{10}{x^2} \right]_{x=10}^{x \to \infty} = \frac{9}{5} + \frac{1}{10} = 1.9 \end{aligned}$$

Example

Consider the jointly distributed random variables $(X, Y) \in [0, \infty) \times [-2, 2]$ whose density is the function $f(x, y) = 1/(4e^x)$. Find the mean of $X + Y$.

Solution

$$
\begin{aligned}
\mathbb{E}(X+Y) &= \int_0^\infty \int_{-2}^2 (x+y)\left(\frac{1}{4e^x}\right) dy\, dx \\
&= \int_0^\infty \frac{1}{4} e^{-x} \int_{-2}^2 (x+y)\, dy\, dx \\
&= \int_0^\infty \frac{1}{4} e^{-x} (4x)\, dx \\
&= \int_0^\infty x\, e^{-x}\, dx \\
&= \lim_{M\to\infty} \int_0^M x\, e^{-x}\, dx \\
&= \lim_{M\to\infty} (1 - Me^{-M} - e^{-M}) \\
&= 1
\end{aligned}
$$

Properties of the Expected Value

Theorem

If $X_1, X_2, \ldots, X_k$ are continuous random variables with joint probability density $f(x_1, x_2, \ldots, x_k)$ then
$\mathbb{E}(X_1 + X_2 + \cdots + X_k) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_k)$.

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Theorem

Let X_1, X_2, …, X_k be pairwise independent random variables with joint density $f(x_1, x_2, \ldots, x_k)$, then

$$\mathbb{E}(X_1 X_2 \cdots X_k) = \mathbb{E}(X_1)\,\mathbb{E}(X_2) \cdots \mathbb{E}(X_k).$$

Variance and Standard Deviation

Definition

If X is a continuously distributed random variable with probability density function $f(x)$, the **variance** of X is defined as

$$\mathrm{Var}(X) = \mathbb{E}\left((X-\mu)^2\right) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx,$$

where $\mu = \mathbb{E}(X)$. The **standard deviation** of X is $\sigma(X) = \sqrt{\mathrm{Var}(X)}$.

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Theorem

Let X be a random variable with probability density f and mean μ, then $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mu^2$.

Example

Suppose X is continuously distributed on $[0, \infty)$ with probability density function $f(x) = e^{-x}$. Find $\mathrm{Var}\,(X)$.

Example

Suppose X is continuously distributed on $[0,\infty)$ with probability density function $f(x) = e^{-x}$. Find $\mathrm{Var}(X)$.

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2 \\
&= \int_0^\infty x^2 e^{-x}\,dx - \left(\int_0^\infty x\,e^{-x}\,dx\right)^2 \\
&= 2 - \left(\int_0^\infty x\,e^{-x}\,dx\right)^2 \\
&= 2 - (1)^2 \\
&= 1
\end{aligned}
$$

Example

Suppose $\alpha > 0$ and $x_0 > 0$ and that random variable X has distribution $f_X(x) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}}$ for $x > x_0$.

1. Verify that $f_X(x)$ is a valid probability distribution.
2. Find $\mathbb{E}(X)$.
3. Find $\mathrm{Var}(X)$.

Solution

We can confirm $f_X(x)$ is a valid probability distribution since

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{x_0}^{\infty} \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}}\,dx = \left[-\left(\frac{x_0}{x}\right)^{\alpha}\right]_{x=x_0}^{x\to\infty} = 1.$$

Properties of Variance

Theorem

Let X be a continuous random variable with probability density $f(x)$ and let $a, b \in \mathbb{R}$, then

$$\mathrm{Var}\left(aX + b\right) = a^2\,\mathrm{Var}\left(X\right).$$

Properties of Variance

Theorem

Let X be a continuous random variable with probability density $f(x)$ and let $a, b \in \mathbb{R}$, then

$$\mathrm{Var}\left(a\,X + b\right) = a^2\,\mathrm{Var}\left(X\right).$$

Theorem

Let $X_1, X_2, \ldots, X_k$ be pairwise independent continuous random variables with joint probability density $f(x_1, x_2, \ldots, x_k)$, then

$$\mathrm{Var}\left(X_1 + X_2 + \cdots + X_k\right) = \mathrm{Var}\left(X_1\right) + \mathrm{Var}\left(X_2\right) + \cdots + \mathrm{Var}\left(X_k\right).$$

Homework

- Read Sections
- Exercises:

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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