Discrete Random Variables

An Undergraduate Introduction to Financial Mathematics

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Objectives

In this lesson we will learn:

- the definition of a discrete random variable,
- describe some common types of discrete random variables,
- define the probability mass function or a discrete random variable,
- describe some properties of the probability mass function.

Random Variables

Definition

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$$X \in \{0, 1, \ldots, N\}$$

Remark: the value of X is unknown until the experiment is conducted.

Probability Mass Functions

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A **probability mass function** (or **probability distribution**) is a function assigning a probability to each element in the sample space Ω. If the random variable is denoted X, then the probability mass function will be denoted $f_X(x)$ and

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If $\Omega = \{x_1, x_2, \ldots, x_N\}$ and f_X is a probability mass function then:

1. $0 \leq f_X(x_i) \leq 1$ for $i = 1, 2, \ldots, N$, and
2. $1 = \sum_{i=1}^{N} f_X(x_i)$.

Example

Let X be a discrete random variable with $\Omega = \{0, 1, 2, 3, \ldots\}$.
Suppose that

$$\begin{aligned} f_X(0) &= f_X(1) \\ f_X(k+1) &= \frac{1}{k} f_X(k) \text{ for } k \in \mathbb{N}. \end{aligned}$$

Find $f_X(0) = \mathbb{P}(X = 0)$.

Solution

Let $p = f_X(0) = f_X(1)$, then

$$\begin{aligned} f_X(2) &= \frac{1}{1} f_X(1) = p \\ f_X(3) &= \frac{1}{2} f_X(2) = \frac{1}{2!} p \\ f_X(4) &= \frac{1}{3} f_X(3) = \frac{1}{3!} p \\ &\vdots \\ f_X(k+1) &= \frac{1}{k} f_X(k) = \frac{1}{k!} p. \end{aligned}$$

Since

$$\begin{aligned} 1 &= \sum_{k=0}^{\infty} f_X(k) = p + p + p + \frac{p}{2!} + \frac{p}{3!} + \cdots \\ &= 2p + p \sum_{k=1}^{\infty} \frac{1}{k!} = 2p + p(e-1) \\ p &= \frac{1}{1+e}. \end{aligned}$$

Bernoulli and Binomial Random Variables (1 of 2)

Definition

A **Bernoulli** random variable can be thought of as having sample space $\Omega = \{0, 1\}$ and probability function

$$f_X(x) = \begin{cases} p & \text{if } x = 1, \\ 1 - p & \text{if } x = 0. \end{cases}$$

The 0 outcome is often termed "failure" while the 1 outcome is called "success".

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Definition

A **binomial** random variable is the number of successes out of n independent Bernoulli experiments. The number of trials n is fixed and the Bernoulli probabilities remain fixed between trials.

Binomial Random Variables (2 of 2)

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Thus $\mathbb{P}(X = x)$ is given by

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}.$$

Example (1 of 2)

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What is the probability that in a family of four children, exactly three of them are male?

Assuming the genders of the children are independent and equally likely to be male or female, the probability of the desired outcome is the binomial probability,

$$\mathbb{P}\left(\text{exactly 3 of 4 children male}\right) = \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Example (2 of 2)

The probability that a computer memory chip is defective is 0.02. A SIMM (single in-line memory module) contains 16 chips for data storage and a 17th chip for error correction. The SIMM can operate correctly if one chip is defective, but not if two or more are defective. What is the probability that the SIMM functions correctly?

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The probability that a computer memory chip is defective is 0.02. A SIMM (single in-line memory module) contains 16 chips for data storage and a 17th chip for error correction. The SIMM can operate correctly if one chip is defective, but not if two or more are defective. What is the probability that the SIMM functions correctly?

The SIMM will function correctly if there is at most one malfunctioning memory chip.

$$
\begin{aligned}
&\mathbb{P}\,(\text{no bad chips} \cup \text{one bad chip}) \\
&\quad = \mathbb{P}\,(\text{no bad chips}) + \mathbb{P}\,(\text{one bad chip}) \\
&\quad = (1 - 0.02)^{17} + \binom{17}{1}(0.02)(1 - 0.02)^{16} \\
&\quad \approx 0.955413
\end{aligned}
$$

Example

Suppose you toss a coin repeatedly until the result is heads. Let X be the random variable representing the number of times the coin flips until the first head appears. Find an expression for $\mathbb{P}(X = n) = f_X(n)$ for $n \in \mathbb{N}$.

Solution

Assume all coin flips are independent and the probability of heads on an individual flip is $0 < p < 1$.

$$\begin{aligned} f_X(1) &= p \\ f_X(2) &= (1-p)p \\ f_X(3) &= (1-p)^2 p \\ &\vdots \\ f_X(n) &= (1-p)^{n-1} p \end{aligned}$$

for $n \in \mathbb{N}$.

Note: the random variable X with this probability mass function is called a **geometric random variable**.

Homework

- Read Sections 2.4, 2.5
- Exercises: 13–15, 17

Credits

These slides are adapted from the textbook,

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