Primal and Dual Programs

MATH 472 *Financial Mathematics*

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2018

Objectives

In this lesson we will learn:

- to form the dual linear program to any primary linear program,
- properties of the dual linear program,
- relationships between the optimal solutions to the primal and dual linear programs.

Dual Problems

For every linear programming problem of the type discussed earlier, there is an associated problem known as its **dual.** Henceforth the original problem will be known as the **primal.** These paired optimization problems are related in the following ways.

Primal: Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Dual: Minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

Observations

Primal: Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Dual: Minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

Note:

1. the process of maximization in the primal is replaced with the process of minimization in the dual,
2. the unknown of the dual is a vector $\mathbf{y}$ with m components,
3. the vector $\mathbf{b}$ moves from the constraint of the primal to the cost function of the dual,
4. the vector $\mathbf{c}$ moves from the cost of the primal to the constraint of the dual,
5. the constraints of the dual are inequalities and there are n of them.

Example

Find the dual of the linear program: maximize $3x_1 + 2x_2$ subject to $\langle x_1, x_2 \rangle \geq \mathbf{0}$ and

$$\begin{aligned} -2x_1 + x_2 &\leq 2 \\ x_1 + 2x_2 &\leq 8. \end{aligned}$$

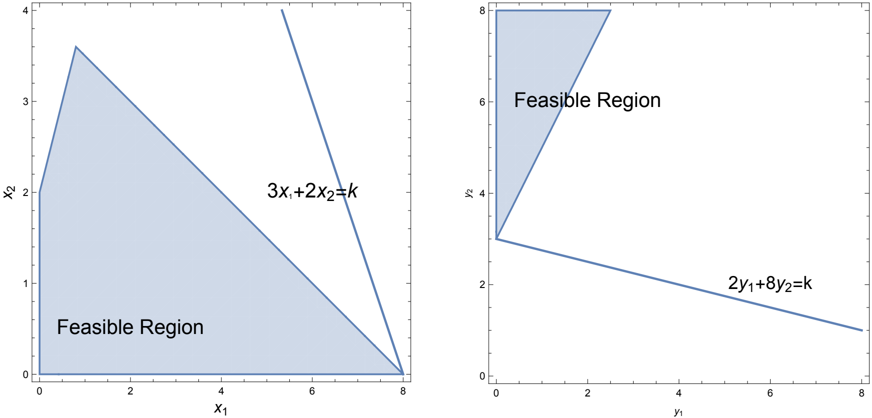
Sketch the feasible sets for the primal and dual problems and solve the primal and dual problems.

Solution: Dual Program

Dual: minimize $2y_1 + 8y_2$ subject to $\langle y_1, y_2 \rangle \geq \mathbf{0}$ and

$$\begin{aligned} -2y_1 + y_2 &\geq 3 \\ y_1 + 2y_2 &\geq 2. \end{aligned}$$

Feasible Sets



The primal problem is maximized at $\langle x_1, x_2\rangle = \langle 8, 0\rangle$. The dual problem is minimized at $\langle y_1, y_2\rangle = \langle 0, 3\rangle$.

Equivalences and Duals (1 of 5)

General Linear Program

Maximize $\mathbf{c}^T\mathbf{x}$ subject to

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ \hat{A}\mathbf{x} &\geq \hat{\mathbf{b}} \\ \tilde{A}\mathbf{x} &= \tilde{\mathbf{b}}. \end{aligned}$$

Symmetric Linear Program

Maximize $\langle \mathbf{c}, -\mathbf{c} \rangle^T \langle \mathbf{x}^+, \mathbf{x}^- \rangle$ subject to

$$\begin{bmatrix} A & -A \\ -\hat{A} & \hat{A} \\ \tilde{A} & -\tilde{A} \\ -\tilde{A} & \tilde{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\hat{\mathbf{b}} \\ \tilde{\mathbf{b}} \\ -\tilde{\mathbf{b}} \end{bmatrix}.$$

Equivalences and Duals (1 of 5)

General Linear Program

Maximize $\mathbf{c}^T\mathbf{x}$ subject to

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ \hat{A}\mathbf{x} &\geq \hat{\mathbf{b}} \\ \tilde{A}\mathbf{x} &= \tilde{\mathbf{b}}. \end{aligned}$$

Symmetric Linear Program

Maximize $\langle \mathbf{c}, -\mathbf{c}\rangle^T \langle \mathbf{x}^+, \mathbf{x}^- \rangle$ subject to

$$\begin{bmatrix} A & -A \\ -\hat{A} & \hat{A} \\ \tilde{A} & -\tilde{A} \\ -\tilde{A} & \tilde{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\hat{\mathbf{b}} \\ \tilde{\mathbf{b}} \\ -\tilde{\mathbf{b}} \end{bmatrix}.$$

Now formulate the dual of the Symmetric Linear Program.

Equivalences and Duals (2 of 5)

Dual: minimize $\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$ subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T & -\tilde{A}^T \\ -A^T & \hat{A}^T & -\tilde{A}^T & \tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}}^+ \\ \tilde{\mathbf{y}}^- \end{bmatrix} \geq \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \end{bmatrix},$$

with $\mathbf{y} \geq \mathbf{0}$, $\hat{\mathbf{y}} \geq \mathbf{0}$, $\tilde{\mathbf{y}}^+ \geq \mathbf{0}$, and $\tilde{\mathbf{y}}^- \geq \mathbf{0}$.

Equivalences and Duals (3 of 5)

Let $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}^+ - \tilde{\mathbf{y}}^-$ and then $\tilde{\mathbf{y}}$ is unrestricted in sign and the dual problem can be restated as

minimize $\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$ subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \\ -A^T & \hat{A}^T & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} \geq \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \end{bmatrix}.$$

Equivalences and Duals (3 of 5)

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minimize $\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$ subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \\ -A^T & \hat{A}^T & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} \geq \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \end{bmatrix}.$$

Remark: the constraints are

$$\begin{aligned} A^T\mathbf{y} - \hat{A}^T\hat{\mathbf{y}} + \tilde{A}^T\tilde{\mathbf{y}} &\geq \mathbf{c} \\ -A^T\mathbf{y} + \hat{A}^T\hat{\mathbf{y}} - \tilde{A}^T\tilde{\mathbf{y}} &\geq -\mathbf{c} \end{aligned}$$

Equivalences and Duals (3 of 5)

Let $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}^+ - \tilde{\mathbf{y}}^-$ and then $\tilde{\mathbf{y}}$ is unrestricted in sign and the dual problem can be restated as

minimize $\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$ subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \\ -A^T & \hat{A}^T & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} \geq \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \end{bmatrix}.$$

Remark: the constraints are

$$\begin{aligned} A^T\mathbf{y} - \hat{A}^T\hat{\mathbf{y}} + \tilde{A}^T\tilde{\mathbf{y}} &\geq \mathbf{c} \\ -A^T\mathbf{y} + \hat{A}^T\hat{\mathbf{y}} - \tilde{A}^T\tilde{\mathbf{y}} &\geq -\mathbf{c} \\ \text{which implies} & \\ A^T\mathbf{y} - \hat{A}^T\hat{\mathbf{y}} + \tilde{A}^T\tilde{\mathbf{y}} &= \mathbf{c} \end{aligned}$$

Equivalences and Duals (4 of 5)

Symmetric Linear Program

Maximize $\langle \mathbf{c}, -\mathbf{c} \rangle^T \langle \mathbf{x}^+, \mathbf{x}^- \rangle$
subject to

$$\begin{bmatrix} A & -A \\ -\hat{A} & \hat{A} \\ \tilde{A} & -\tilde{A} \\ -\tilde{A} & \tilde{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\hat{\mathbf{b}} \\ \tilde{\mathbf{b}} \\ -\tilde{\mathbf{b}} \end{bmatrix}.$$

Dual Linear Program

Minimize
$\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$
subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} = \mathbf{c}.$$

Equivalences and Duals (5 of 5)

Dual Linear Program

Minimize
$\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$
subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} = \mathbf{c}.$$

Primal Linear Program

Maximize $\mathbf{c}^T\mathbf{x}$ subject to

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ \hat{A}\mathbf{x} &\geq \hat{\mathbf{b}} \\ \tilde{A}\mathbf{x} &= \tilde{\mathbf{b}}. \end{aligned}$$

Equivalences and Duals (5 of 5)

Primal Linear Program

Maximize $\mathbf{c}^T\mathbf{x}$ subject to

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ \hat{A}\mathbf{x} &\geq \hat{\mathbf{b}} \\ \tilde{A}\mathbf{x} &= \tilde{\mathbf{b}}. \end{aligned}$$

Dual Linear Program

Minimize
$\langle \mathbf{b}, -\hat{\mathbf{b}}, \tilde{\mathbf{b}}, -\tilde{\mathbf{b}} \rangle^T \langle \mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}^+, \tilde{\mathbf{y}}^- \rangle$
subject to

$$\begin{bmatrix} A^T & -\hat{A}^T & \tilde{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \\ \tilde{\mathbf{y}} \end{bmatrix} = \mathbf{c}.$$

Remark: unrestricted decision variables in the primal (dual) problem induce equality constraints in the dual (primal) problem.

Dual of the Dual

Theorem

The dual of the dual is the primal.

Proof

Starting with the dual problem,

$$\text{Minimize } \mathbf{b}^T\mathbf{y} \quad \text{subject to } A^T\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0}.$$

We can re-write the dual in general form,

$$\text{Maximize } (-\mathbf{b})^T\mathbf{y} \quad \text{subject to } (-A)^T\mathbf{y} \leq -\mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0}.$$

Now the dual of this problem (*i.e.*, the dual of the dual) is

$$\text{Minimize } (-\mathbf{c})^T\mathbf{x} \quad \text{subject to } ((-A)^T)^T\mathbf{x} \geq -\mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}.$$

This problem is logically equivalent to the problem

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0},$$

which is the primal problem.

Weak Duality Theorem

Theorem (Weak Duality Theorem)

If $\mathbf{x}$ *and* $\mathbf{y}$ *are the feasible solutions of the primal and dual problems respectively, then* $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$*. If* $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$ *then these solutions are optimal for their respective problems.*

Proof of Weak Duality Theorem

Feasible solutions to the primal and the dual problems must satisfy the constraints $A\mathbf{x} \leq \mathbf{b}$ with $\mathbf{x} \geq 0$ (for the primal problem) and $A^T\mathbf{y} \geq \mathbf{c}$ with $\mathbf{y} \geq \mathbf{0}$ (for the dual). Multiply the constraint in the dual by $\mathbf{x}^T$

$$\mathbf{x}^T A^T \mathbf{y} \geq \mathbf{x}^T \mathbf{c} \iff \mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T A \mathbf{x}.$$

Multiply the constraint in the primal by $\mathbf{y}^T$

$$\mathbf{y}^T A \mathbf{x} \leq \mathbf{y}^T \mathbf{b} = \mathbf{b}^T \mathbf{y}.$$

Directions of the inequalities are preserved because $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$. Combining these last two inequalities produces

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T A \mathbf{x} \leq \mathbf{b}^T \mathbf{y}.$$

Therefore we have $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$. If $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$ then $\mathbf{x}$ and $\mathbf{y}$ must be optimal since no $\mathbf{x}$ can make $\mathbf{c}^T\mathbf{x}$ larger than $\mathbf{b}^T\mathbf{y}$ and no $\mathbf{y}$ can make $\mathbf{b}^T\mathbf{y}$ smaller than $\mathbf{c}^T\mathbf{x}$.

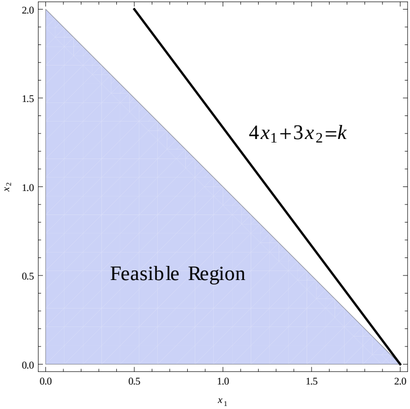
Example (1 of 2)

Primal: Maximize $4x_1 + 3x_2$ subject to $x_1 + x_2 \leq 2$ and $x_1, x_2 \geq 0$.

Dual: Minimize $2y_1$ subject to $y_1 \geq 3$, $y_1 \geq 4$, and $y_1 \geq 0$.

Example (2 of 2)

The minimum value of y_1 subject to the constraints must be $y_1 = 4$. According to the Weak Duality Theorem then the minimum of the cost function of the primal must be at least $2y_1 = 8$. Applying the level set argument as before, the largest value of k for which the level set $4x_1 + 3x_2 = k$ intersects the set of feasible points for the primal is $k = 8$.



Existence Theorem

Theorem (Existence)

The primal and the dual problems have optimal solutions if and only if both problems have nonempty feasible sets of solution vectors.

Existence Theorem

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The primal and the dual problems have optimal solutions if and only if both problems have nonempty feasible sets of solution vectors.

Proof.

Suppose the primal and the dual problems have optimal solutions, then both problems have nonempty feasible sets.

Suppose the primal and the dual problems have nonempty feasible sets. Let $\mathbf{y}$ be any feasible vector of the dual, then by the Weak Duality Theorem, $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$ for all feasible vectors of the primal problem. The objective function $\mathbf{c}^T\mathbf{x}$ is a continuous function defined on a closed set and is bounded above, thus it has a maximum.

A similar argument shows the dual has a minimum. □

Complementary Slackness

Theorem

*If **x** and **y** are feasible solutions to the primal and dual problems respectively, they are optimal if and only if*

$$x_j = 0 \quad \textit{whenever} \quad (A^T\mathbf{y})_j = \sum_{i=1}^{m} A_{ij} y_i > c_j$$

$$y_i = 0 \quad \textit{whenever} \quad (A\mathbf{x})_j = \sum_{j=1}^{n} A_{ij} x_j < b_j.$$

Optimality in the primal and dual problems requires the kth variable in one problem to be zero whenever the kth constraint in the other problem is ineffective.

Proof (1 of 3)

Suppose $\mathbf{x}$ and $\mathbf{y}$ are optimal for their respective problems and that $\mathbf{b}^T\mathbf{y} = \mathbf{c}^T\mathbf{x}$. From the Weak Duality Theorem we have

$$\begin{aligned} \mathbf{b}^T\mathbf{y} = \mathbf{y}^T A\mathbf{x} &= \mathbf{c}^T\mathbf{x} \\ (\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} &= 0 \\ (A^T\mathbf{y} - \mathbf{c})^T\mathbf{x} &= 0. \end{aligned}$$

Since $\mathbf{x} \geq \mathbf{0}$ and $A^T\mathbf{y} \geq \mathbf{c}$ then vector $\mathbf{x}$ must be zero in every component for which vector $A^T\mathbf{y} - \mathbf{c}$ is positive and vice versa.

Proof (2 of 3)

Suppose that **x** and **y** are feasible solutions for the primal and dual problems respectively and

$$x_j = 0 \quad \text{whenever} \quad \sum_{i=1}^{m} A_{ij} y_i > c_j$$

$$y_i = 0 \quad \text{whenever} \quad \sum_{j=1}^{n} A_{ij} x_j < b_j.$$

This implies

$$\begin{aligned} \sum_{j=1}^{n} \left(\left[\sum_{i=1}^{m} A_{ij} y_i \right] - c_j \right) x_j &= 0 \\ \sum_{j=1}^{n} \sum_{i=1}^{m} A_{ij} y_i x_j = \sum_{i=1}^{m} y_i \sum_{j=1}^{n} A_{ij} x_j &= \sum_{j=1}^{n} c_j x_j \\ \mathbf{y}^T A \mathbf{x} &= \mathbf{c}^T \mathbf{x}. \end{aligned}$$

Proof (3 of 3)

A similar argument shows $\mathbf{b}^T\mathbf{y} = \mathbf{y}^T A\mathbf{x}$.

Hence we have the equation,

$$\mathbf{b}^T\mathbf{y} = \mathbf{y}^T A\mathbf{x} = \mathbf{c}^T\mathbf{x}.$$

By the Weak Duality Theorem, $\mathbf{x}$ and $\mathbf{y}$ must be optimal for the primal and dual problems respectively.

Example (1 of 4)

Primal: Maximize $\mathbf{c}^T\mathbf{x} = -3x_1 + 2x_2 - x_3 + 3x_4$ subject to $\mathbf{x} \geq \mathbf{0}$ and

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Example (2 of 4)

Dual: Minimize $\mathbf{b}^T\mathbf{y} = 5y_1 + 3y_2$ subject to

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} -3 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

Example (2 of 4)

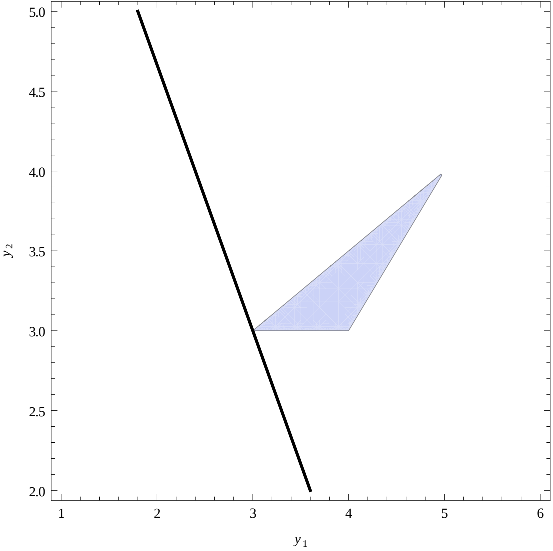
Dual: Minimize $\mathbf{b}^T\mathbf{y} = 5y_1 + 3y_2$ subject to

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} -3 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

Expressed as a system of inequalities, these constraints are

$$\begin{aligned} y_1 - 2y_2 &\geq -3 \\ y_1 &\geq 2 \\ -y_1 + y_2 &\geq -1 \\ y_2 &\geq 3. \end{aligned}$$

Example (3 of 4)



Optimal solution is at $(y_1, y_2) = (3, 3)$ and has value 24.

Example (4 of 4)

Strict inequality is present in the second and third constraints since

$$\begin{aligned} y_1 = 3 &> 2 \\ -y_1 + y_2 = 0 &> -1. \end{aligned}$$

Thus the second and third components of $\mathbf{x}$ in the primal problem must be zero. Therefore the primal can be recast as
Primal: Maximize $-3x_1 + 3x_4$ subject to $x_1 \geq 0$, $x_4 \geq 0$ and

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 + x_4 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Thus $x_1 = 5$ and $x_4 = 13$, the maximum of the cost function for the primal is 24 and it occurs at $(x_1, x_2, x_3, x_4) = (5, 0, 0, 13)$.

Duality Theorem

Theorem (Duality Theorem)

One and only one of the following four cases can be true.

1. *There exist optimal solutions for both the primal and dual problems and the maximum of* $\mathbf{c}^T\mathbf{x}$ *equals the minimum of* $\mathbf{b}^T\mathbf{y}$.
2. *There exists no feasible solution to the primal problem and the dual problem has feasible solutions for which the minimum of* $\mathbf{b}^T\mathbf{y}$ *approaches* $-\infty$.
3. *There exists no feasible solution to the dual problem and the primal problem has feasible solutions for which the maximum of* $\mathbf{c}^T\mathbf{x}$ *approaches* ∞.
4. *Neither the primal nor the dual problem has a feasible solution.*

Farkas Alternative

Remark: before proving the Duality Theorem we must state a lemma which will be used in the proof.

Lemma (Farkas Alternative)

Exactly one of the following two statements is true. Either

1. $A\mathbf{x} \leq \mathbf{b}$ *has a solution* $\mathbf{x} \geq \mathbf{0}$, *or*
2. $A^T\mathbf{y} \geq \mathbf{0}$ *with* $\mathbf{b}^T\mathbf{y} < 0$ *has a solution* $\mathbf{y} \geq \mathbf{0}$.

Proof (1 of 8)

Primal: Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Dual: Minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

Assuming there are feasible solutions to each problem then we can re-write the constraint of the dual as $(-A)^T\mathbf{y} \leq -\mathbf{c}$ with $\mathbf{y} \geq \mathbf{0}$. Thus according to the constraint on the primal, the re-written constraint on the dual, and the conclusion of the Weak Duality Theorem the following inequalities hold for $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$.

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ (-A)^T\mathbf{y} &\leq -\mathbf{c} \\ \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} &\leq \mathbf{0} \end{aligned}$$

Proof (1 of 8)

Primal: Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Dual: Minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

Assuming there are feasible solutions to each problem then we can re-write the constraint of the dual as $(-A)^T\mathbf{y} \leq -\mathbf{c}$ with $\mathbf{y} \geq \mathbf{0}$. Thus according to the constraint on the primal, the re-written constraint on the dual, and the conclusion of the Weak Duality Theorem the following inequalities hold for $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$.

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ (-A)^T\mathbf{y} &\leq -\mathbf{c} \\ \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} &\leq \mathbf{0} \end{aligned}$$

Remark: if equality holds in the last inequality, then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions.

Proof (2 of 8)

These inequalities can be written in the block matrix form

$$\begin{bmatrix} A & 0 \\ 0 & -A^T \\ \mathbf{c}^T & -\mathbf{b}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \\ 0 \end{bmatrix}.$$

According to the Farkas Alternative Lemma either this inequality has a solution $\langle \mathbf{x}, \mathbf{y} \rangle \geq \mathbf{0}$ or the alternative

$$\begin{bmatrix} A^T & 0 & \mathbf{c} \\ 0 & -A & -\mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \lambda \end{bmatrix} \geq \mathbf{0} \text{ and } \begin{bmatrix} \mathbf{b}^T & -\mathbf{c}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \lambda \end{bmatrix} < 0$$

has a solution $\langle \mathbf{u}, \mathbf{v}, \lambda \rangle \geq \mathbf{0}$.

Proof (3 of 8)

We may decompose this block matrix to derive the following system of inequalities:

$$A^T\mathbf{u} + \lambda\mathbf{c} \geq \mathbf{0}, \quad -A\mathbf{v} - \lambda\mathbf{b} \geq \mathbf{0}, \quad \mathbf{b}^T\mathbf{u} - \mathbf{c}^T\mathbf{v} < 0$$

with $\mathbf{u} \geq \mathbf{0}$, $\mathbf{v} \geq \mathbf{0}$, and $\lambda \geq 0$. If $\lambda > 0$ then this system of inequalities is equivalent to the following system.

$$\begin{aligned} A\left(\frac{1}{\lambda}\mathbf{v}\right) &\leq -\mathbf{b} \\ A^T\left(\frac{1}{\lambda}\mathbf{u}\right) &\geq -\mathbf{c} \\ -\mathbf{b}^T\left(\frac{1}{\lambda}\mathbf{u}\right) &> -\mathbf{c}^T\left(\frac{1}{\lambda}\mathbf{v}\right) \end{aligned}$$

Since $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$ the vectors $\frac{1}{\lambda}\mathbf{u} \geq \mathbf{0}$ and $\frac{1}{\lambda}\mathbf{v} \geq \mathbf{0}$ as well.

Proof (4 of 8)

The first two inequalities form a primal problem and its dual.

$$\begin{aligned} \text{Primal:} \quad A\left(\frac{1}{\lambda}\mathbf{v}\right) &\leq -\mathbf{b} \\ \text{Dual:} \quad A^T\left(\frac{1}{\lambda}\mathbf{u}\right) &\geq -\mathbf{c} \end{aligned}$$

If we apply the Weak Duality Theorem, then it must be the case that $-\mathbf{b}^T\left(\frac{1}{\lambda}\mathbf{u}\right) \leq -\mathbf{c}^T\left(\frac{1}{\lambda}\mathbf{v}\right)$, contradicting the inequality:

$$-\mathbf{b}^T\left(\frac{1}{\lambda}\mathbf{u}\right) > -\mathbf{c}^T\left(\frac{1}{\lambda}\mathbf{v}\right)$$

Therefore we know that $\lambda = 0$.

Proof (5 of 8)

Thus the Farkas Alternative simplifies to the following system:

$$A\mathbf{v} \leq \mathbf{0}, \quad A^T\mathbf{u} \geq \mathbf{0}, \quad \text{and} \quad \mathbf{b}^T\mathbf{u} < \mathbf{c}^T\mathbf{v}$$

where $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$. The last inequality implies that $\mathbf{b}^T\mathbf{u} < 0$ or $\mathbf{c}^T\mathbf{v} > 0$. If $\mathbf{b}^T\mathbf{u} < 0$ then the primal problem $A\mathbf{x} \leq \mathbf{b}$ has no feasible solution $\mathbf{x} \geq \mathbf{0}$. To see this note that together the inequalities $\mathbf{x} \geq \mathbf{0}$, $A\mathbf{x} \leq \mathbf{b}$, and $\mathbf{b}^T\mathbf{u} < 0$ imply that

$$\begin{aligned} (A\mathbf{x})^T &\leq \mathbf{b}^T \\ \mathbf{x}^T A^T &\leq \mathbf{b}^T \\ \mathbf{x}^T \left(A^T\mathbf{u}\right) &\leq \mathbf{b}^T\mathbf{u} < 0. \end{aligned}$$

However, $\mathbf{x} \geq \mathbf{0}$ and $A^T\mathbf{u} \geq \mathbf{0}$ and thus $\mathbf{x}^T\left(A^T\mathbf{u}\right) \geq 0$, a contradiction.

Proof (6 of 8)

We may conclude that if $\mathbf{b}^T\mathbf{u} < 0$ then the primal problem has no feasible solution.

If the dual problem also lacks a feasible solution then we are in the fourth case of the theorem.

If the dual problem possesses a feasible solution $\mathbf{y}$, then

$$A^T\mathbf{y} + A^T\lambda\mathbf{u} = A^T(\mathbf{y} + \lambda\mathbf{u}) \geq \mathbf{c}.$$

Since $\mathbf{y} + \lambda\mathbf{u} \geq \mathbf{0}$ for all $\lambda \geq 0$, then $\mathbf{y} + \lambda\mathbf{u}$ is a feasible solution to the dual problem, and

$$\lim_{\lambda\to\infty} \mathbf{b}^T(\mathbf{y} + \lambda\mathbf{u}) = \mathbf{b}^T\mathbf{y} + \lim_{\lambda\to\infty} (\lambda\mathbf{b}^T\mathbf{u}) = -\infty.$$

Proof (7 of 8)

Returning to the other half of our alternatively, namely $\mathbf{c}^T\mathbf{v} > 0$ and assuming there exists a feasible solution to the dual problem, then we have the following inequalities.

$$\begin{aligned} A^T\mathbf{y} &\geq \mathbf{c} \\ \mathbf{y}^T A &\geq \mathbf{c}^T \\ -\mathbf{y}^T A\mathbf{v} &\geq -\mathbf{c}^T\mathbf{v} < 0 \end{aligned}$$

However, $\mathbf{y} \geq \mathbf{0}$ and $-A\,\mathbf{v} \geq \mathbf{0}$ and thus $-\mathbf{y}^T A\mathbf{v} \geq 0$, a contradiction.

Therefore the dual problem has no feasible solution.

Proof (8 of 8)

If the primal problem has no feasible solution, then we are once again in the fourth case of the theorem.

If the primal problem has a feasible solution $\mathbf{x}$, then

$$A\mathbf{x} + A\lambda\mathbf{v} = A(\mathbf{x} + \lambda\mathbf{v}) \leq \mathbf{b}.$$

Since $\mathbf{x} + \lambda\mathbf{v} \geq \mathbf{0}$ for all $\lambda \geq 0$, then $\mathbf{x} + \lambda\mathbf{v}$ is a feasible solution to the primal problem, and

$$\lim_{\lambda\to\infty} \mathbf{c}^T(\mathbf{x} + \lambda\mathbf{v}) = \mathbf{c}^T\mathbf{x} + \lim_{\lambda\to\infty} (\lambda\mathbf{c}^T\mathbf{v}) = \infty.$$

Homework

- Read Section 4.3
- Exercises: 10–14

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

author: J. Robert Buchanan

publisher: World Scientific Publishing Co. Pte. Ltd.

address: 27 Warren St., Suite 401–402, Hackensack, NJ 07601

ISBN: 978-9814407441