Elementary Discrete Probability

MATH 472 *Financial Mathematics*

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Objectives

In this lesson we will learn:

- the terminology of elementary probability,
- elementary rules of probability, and
- applications of probability to games of chance.

Basic Definitions

Definition

- An **experiment** is any activity that generates an observable outcome.
- The collection of all possible outcomes of an experiment is the **sample space** and is denoted Ω.
- An **event** is an outcome or set of outcomes with a specified property (generally denoted with letters A, B, $\dots$).
- The **probability** of an event is a real number measuring the likelihood of the occurrence of the event (generally denoted $\mathbb{P}(A)$, $\mathbb{P}(B)$, $\dots$).

Probability Measure

Definition

A **probability measure** on a sample space Ω is function defined on the subsets of Ω satisfying:

1. If $A \subset \Omega$ then $\mathbb{P}(A) \geq 0$.
2. If $A \cap B = \emptyset$ then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

3. $\mathbb{P}(\Omega) = 1$.

Discrete Events

For the time being we will assume the outcomes of experiments are **discrete** in the sense that the outcomes will be from a set whose members are isolated from each other by gaps.

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Later, we may think of discrete outcomes as the results of experiments with a finite or at most countable number of outcomes.

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To determine the $\mathbb{P}(A)$ we can:

- take the **empirical approach** and conduct (or at least simulate) the experiment N times, count the number of times x that event A occurred, and estimate $\mathbb{P}(A) = x/N$.
- take the **classical approach** and determine the number of outcomes of the experiment (call this number M), assume the outcomes are equally likely, and determine the number of outcomes y among the M in which event A occurs. Then $\mathbb{P}(A) = y/M$.

Determining the Size of the Sample Space

- Determine the number of outcomes of the experiment of tossing a pair of dice.

- Determine the number of outcomes of the experiment of tossing a pair of dice **twice**.

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$$M' = (36)^2 = 1296$$

Determining the Classical Probability

You and a friend each toss a fair coin. You win if your coin matches your friend's coin.

1. Determine the sample space of the experiment.

2. What is the probability that you win?

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1. Determine the sample space of the experiment.

$$\Omega = \{HH, HT, TH, TT\}$$

2. What is the probability that you win? In 2 out of the 4 equally likely outcomes you win, thus

$$\mathbb{P}(\text{win}) = \frac{2}{4} = \frac{1}{2}.$$

Addition Rule (1 of 2)

Notation: $\mathbb{P}(A \cup B)$ denotes the probability that event A or B occurs.

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Definition

Two events are **mutually exclusive** if they cannot occur together.

Addition Rule (2 of 2)

Theorem (Addition Rule)

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ *where* $\mathbb{P}(A \cap B)$ *is the probability that events* A *and* B *occur together.*

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Corollary

If A *and* B *are mutually exclusive then* $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$*.*

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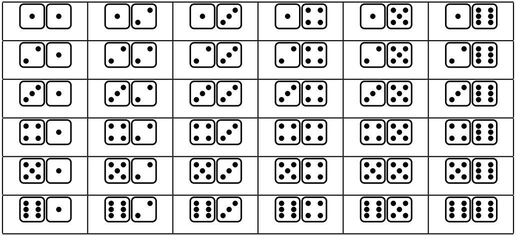
Corollary

If A and B are mutually exclusive then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$*.*

Remark: since A and B are mutually exclusive, $A \cap B$ is an impossible event.

Fair Dice

Find the probability of each outcome of the rolling of a pair of fair dice using the classical method.



Outcomes, Events, and Probabilities

Dice Totals

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

E	$\mathbb{P}(E)$
2	0.0278
3	0.0556
4	0.0833
5	0.1111
6	0.1389
7	0.1667
8	0.1389
9	0.1111
10	0.0833
11	0.0556
12	0.0278

Example

Find the probability that when a pair of fair dice are thrown the total of the dice is less than 6 or odd.

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Solution:

$$\begin{aligned}
\mathbb{P}((X < 6) \cup (X \text{ odd})) &= \mathbb{P}(X < 6) + \mathbb{P}(X \text{ odd}) \\
&\quad - \mathbb{P}((X < 6) \cap (X \text{ odd})) \\
&= \frac{10}{36} + \frac{18}{36} - \frac{6}{36} \\
&= \frac{11}{18}
\end{aligned}$$

Monty Hall Problem

A game show host hides a prize behind one of three doors. A contestant must guess which door hides the prize. First, the contestant announces the door they have chosen. The host will then open one of the two doors, not chosen, in order to reveal the prize is not behind it. The host then tells the contestant they may keep their original choice or switch to the other unopened door. Should the contestant switch doors?

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Remark: you may wish to try a simulator for this game.

Sampling with Replacement

There are three red socks and six black socks in a drawer. What is the probability that in a sample of size 2 taken with replacement, that

1. the sample contains a red sock?
2. the sample contains exactly one red sock?
3. the sample contains two red socks?

Conditional Probability

Definition

The probability that one event occurs given that another event has occurred is called **conditional probability.** The probability that event A occurs given that event B has occurred is denoted $\mathbb{P}(A|B)$.

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What is the contestant's probability of winning if they do not switch doors? Should they switch?

Multiplication Rule

Theorem (Multiplication Rule)

For events A and B, the probability of A and B occurring is

$$\mathbb{P}\left(A \cap B\right) = \mathbb{P}\left(A\right)\mathbb{P}\left(B|A\right),$$

or equivalently

$$\mathbb{P}\left(B|A\right) = \frac{\mathbb{P}\left(A \cap B\right)}{\mathbb{P}\left(A\right)}.$$

provided $\mathbb{P}\left(A\right) > 0$*.*

Example

There are two drawers (labeled α and β). Drawer α contains only red socks. Drawer β contains equals numbers of red socks and black socks. A fair coin is tossed to determine a drawer to select from and a red sock is drawn.

What is the probability drawer α was chosen given that a red sock was drawn?

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There are two drawers (labeled α and β). Drawer α contains only red socks. Drawer β contains equals numbers of red socks and black socks. A fair coin is tossed to determine a drawer to select from and a red sock is drawn.

What is the probability drawer α was chosen given that a red sock was drawn?

$$\mathbb{P}(\alpha \mid \text{red}) = \frac{\mathbb{P}(\alpha \cap \text{red})}{\mathbb{P}(\text{red})} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} = \frac{2}{3}$$

Example: Roulette (1 of 2)

One type of roulette wheel, known as the American type, has 38 potential outcomes represented by the integers 1 through 36 and two special outcomes 0 and 00. The positive integers are placed on alternating red and black backgrounds while 0 and 00 are on green backgrounds.

Example: Roulette (2 of 2)

Question: What is the probability that the outcome is less than 10 and more than 3 given that the outcome is an even number? (0 and 00 are considered even numbers.)

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Solution:

$$\begin{aligned} \mathbb{P}((3 < X < 10) \mid (X \text{ even})) &= \frac{\mathbb{P}((3 < X < 10) \cap (X \text{ even}))}{\mathbb{P}(X \text{ even})} \\ &= \frac{3/38}{20/38} \\ &= \frac{3}{20} \end{aligned}$$

Bayes' Formula

- Suppose A and B are two events, then

$$A = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c).$$

- Since $A \cap B$ and $A \cap B^c$ are mutually exclusive events, then

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}((A \cap B) \cup (A \cap B^c)) \\
&= \mathbb{P}(A \cap B)) + \mathbb{P}(A \cap B^c) \\
&= \mathbb{P}(A|B)\,\mathbb{P}(B) + \mathbb{P}(A|B^c)\,\mathbb{P}(B^c).
\end{aligned}$$

- Use this to determine $\mathbb{P}(B|A)$:

$$\begin{aligned}
\mathbb{P}(B|A) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A|B)\,\mathbb{P}(B) + \mathbb{P}(A|B^c)\,\mathbb{P}(B^c)} \\
&= \frac{\mathbb{P}(A|B)\,\mathbb{P}(B)}{\mathbb{P}(A|B)\,\mathbb{P}(B) + \mathbb{P}(A|B^c)\,\mathbb{P}(B^c)}
\end{aligned}$$

Example

A company uses two machines to produce LCD screens for smartphones. Machine A produces 20% of the total number of LCD screens while machine B produces 80%. The LCDs produced by machine A have a defect rate of 5% while the defect rate of machine B is 1%. Find the probability that a randomly selected LCD screen produced by the company came from machine A given that the screen is defective.

Solution

Take note of the probabilities given in the example:

$$\begin{aligned}\mathbb{P}(A) &= 0.20\\ \mathbb{P}(B) &= 0.80\\ \mathbb{P}(D|A) &= 0.05\\ \mathbb{P}(D|B) &= 0.01.\end{aligned}$$

Using Bayes' formula,

$$\begin{aligned}\mathbb{P}(A|D) &= \frac{\mathbb{P}(A\cap D)}{\mathbb{P}(D)} = \frac{\mathbb{P}(D|A)\,\mathbb{P}(A)}{\mathbb{P}(D|A)\,\mathbb{P}(A)+\mathbb{P}(D|B)\,\mathbb{P}(B)}\\ &= \frac{(0.05)(0.20)}{(0.05)(0.20)+(0.01)(0.80)} \approx 0.5556.\end{aligned}$$

Independent Events

Definition

If the occurrence of event A does not affect the occurrence of event B we say that the events are **independent.**

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

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Corollary

If events A and B are independent then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

Example

Question: what is the probability of two red outcomes on successive spins of the roulette wheel?

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Question: what is the probability of two red outcomes on successive spins of the roulette wheel?
Answer: since the outcomes of spins of the roulette wheel are independent, then

$$\begin{aligned}
\mathbb{P}\,(\text{1st spin red} \cap \text{2nd spin red}) &= \mathbb{P}\,(\text{1st spin red})\,\mathbb{P}\,(\text{2nd spin red}) \\
&= \left(\frac{18}{38}\right)\left(\frac{18}{38}\right) \\
&= \frac{81}{361} \\
&\approx 0.224377.
\end{aligned}$$

Homework

- Read Sections 2.1–2.3
- Exercise: 1–12.

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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