Expected Value and Variance

MATH 472 *Financial Mathematics*

J Robert Buchanan

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Objectives

In this lesson we will learn:

- the definition of expected value,
- how to calculate the expected value of a random variable,
- the properties of expected value,
- the definition of variance,
- the properties of variance.

Expected Value of a Random Variable

Definition

If X is a discrete random variable with probability mass function $f_X(x)$ then the **expected value of** X is denoted $\mathbb{E}(X)$ and defined as

$$\mathbb{E}(X) = \sum_{x \in \Omega} (x \cdot f_X(x)).$$

Example

What is the expected value of the number of female children in a family of 5 children?

Example

What is the expected value of the number of female children in a family of 5 children?

If X represents the number of female children in a family with five children then

$$\begin{aligned}\mathbb{E}(X) &= \sum_{x\in\Omega}(x\cdot f_X(x)) \\ &= \sum_{x=0}^{5}\left(x\cdot\binom{5}{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{5-x}\right) \\ &= \frac{5!}{32}\sum_{x=0}^{5}\frac{x}{(x!)(5-x)!} \\ &= 2.5\end{aligned}$$

Expected Value of a Function of a Random Variable

Definition

If G is a function of the random variable X, then the **expected value of** G is

$$\mathbb{E}\left(G(X)\right) = \sum_{x \in \Omega} G(x) f_X(x).$$

Example

What is the expected value of the square of the number of female children in a family of 5 children?

Example

What is the expected value of the square of the number of female children in a family of 5 children?

If X represents the number of female children in a family with five children then

$$\begin{aligned}
\mathbb{E}\left(X^2\right) &= \sum_{x \in \Omega} \left(x^2 \cdot f_X(x)\right) \\
&= \sum_{x=0}^{5} \left(x^2 \cdot \binom{5}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}\right) \\
&= 7.5
\end{aligned}$$

Linearity of Expected Value

Theorem

If X is a random variable and a is a constant, then $\mathbb{E}(aX) = a\mathbb{E}(X)$.

Linearity of Expected Value

Theorem

If X is a random variable and a is a constant, then $\mathbb{E}(aX) = a\mathbb{E}(X)$.

Proof.

$$\mathbb{E}(aX) = \sum_{x \in \Omega} ((ax) \cdot f_X(x)) = a \sum_{x \in \Omega} (x \cdot f_X(x)) = a\mathbb{E}(X).$$

□

Joint Probability Function

Definition

If X and Y are discrete random variables the **joint probability mass function** of X and Y is denoted $f_{X,Y}(x,y)$ where

$$f_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y) = \mathbb{P}((X = x) \cap (Y = y)).$$

Joint Probability Function

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A joint probability mass function has the properties:

1. $0 \leq f_{X,Y}(x,y) \leq 1$,
2. $\displaystyle\sum_{(x,y)\in\Omega} f_{X,Y}(x,y) = 1$.

Marginal Probability

Definition

If the joint probability mass function of (X, Y) is $f_{X,Y}(x, y)$ then the function

$$f_X(x) = \sum_{y \in \Omega} f_{X,Y}(x, y)$$

is called the **marginal probability distribution of X**.

We may define a **marginal probability distribution for Y** similarly.

Expected Value of a Sum (1 of 2)

Theorem

If $X_1, X_2, \ldots, X_k$ are random variables then

$$\mathbb{E}\left(X_1 + X_2 + \cdots + X_k\right) = \mathbb{E}\left(X_1\right) + \mathbb{E}\left(X_2\right) + \cdots + \mathbb{E}\left(X_k\right).$$

Expected Value of a Sum (2 of 2)

Proof.

If $k = 2$ then

$$\begin{aligned}
\mathbb{E}(X+Y) &= \sum_x \sum_y \left((x+y) f_{X,Y}(x,y)\right) \\
&= \sum_x \sum_y x\, f_{X,Y}(x,y) + \sum_y \sum_x y\, f_{X,Y}(x,y) \\
&= \sum_x x \sum_y f_{X,Y}(x,y) + \sum_y y \sum_x f_{X,Y}(x,y) \\
&= \sum_x x\, f_X(x) + \sum_y y\, f_Y(y) \\
&= \mathbb{E}(X) + \mathbb{E}(Y)\,.
\end{aligned}$$

The general case then holds by induction on k. □

Expected Value of a Sum of Functions

Corollary

Let X_1, X_2, ..., X_k be random variables and let F_i be a function of X_i for $i = 1, 2, \ldots, k$ then

$$\mathbb{E}\left(F_1(X_1) + \cdots + F_k(X_k)\right) = \mathbb{E}\left(F_1(X_1)\right) + \cdots + \mathbb{E}\left(F_k(X_k)\right).$$

Expected Value of Binomial Random Variable

If X_i is a Bernoulli random variable then

$$\mathbb{E}(X_i) = (1)(p) + (0)(1 - p) = p.$$

Expected Value of Binomial Random Variable

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If the binomial random variable $X = X_1 + \cdots + X_n$ where each X_i is a Bernoulli random variable, then

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) = n\,p.$$

Expected Value of a Geometric RV

- The probability mass function of a geometric RV is given by

$$\mathbb{P}(X = n) = f_X(n) = (1-p)^{n-1}p \text{ for } n \in \mathbb{N}.$$

- Recall the geometric series summation,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ for } |z| < 1.$$

- Differentiate the summation and replace z by $1 - p$.

Expected Value of a Geometric RV

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- Differentiate the summation and replace z by $1 - p$.

$$\begin{aligned}
\sum_{n=1}^{\infty} n\, z^{n-1} &= \frac{1}{(1-z)^2} \\
\sum_{n=1}^{\infty} n\,(1-p)^{n-1} p &= \frac{p}{(1-(1-p))^2} \\
\mathbb{E}(X) &= \frac{1}{p}
\end{aligned}$$

Independent Random Variables (1 of 2)

If X and Y are **independent** random variables then

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

where $f_X(x)$ is the marginal probability distribution of X and $f_Y(y)$ is the marginal probability distribution of Y.

Independent Random Variables (1 of 2)

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$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

where $f_X(x)$ is the marginal probability distribution of X and $f_Y(y)$ is the marginal probability distribution of Y.

Theorem

Let $X_1, X_2, \ldots, X_k$ be pairwise independent random variables, then

$$\mathbb{E}\left(X_1 X_2 \cdots X_k\right) = \mathbb{E}\left(X_1\right)\mathbb{E}\left(X_2\right)\cdots\mathbb{E}\left(X_k\right).$$

Independent Random Variables (2 of 2)

Proof.

Now let X and Y be independent random variables with joint probability distribution $f_{X,Y}(x, y)$.

$$\begin{aligned} \mathbb{E}(X\,Y) &= \sum_{(x,y)\in\Omega} x\,y\,f_{X,Y}(x,y) \\ &= \sum_x \sum_y x\,y\,f_X(x) f_Y(y) \\ &= \sum_x x\,f_X(x) \sum_y y\,f_Y(y) \\ &= \mathbb{E}(X)\,\mathbb{E}(Y) \end{aligned}$$

The general case holds by induction on k. □

Variance (1 of 2)

Definition

If X is a random variable, the **variance** of X is denoted $\mathrm{Var}(X)$ and

$$\mathrm{Var}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right).$$

The **standard deviation** of X is denoted $\sigma(X) = \sqrt{\mathrm{Var}(X)}$.

Variance (2 of 2)

Theorem

Let X be a random variable, then the variance of X is

$$\mathrm{Var}\left(X\right) = \mathbb{E}\left(X^2\right) - \mathbb{E}\left(X\right)^2 .$$

Variance (2 of 2)

Theorem

Let X be a random variable, then the variance of X is

$$\mathrm{Var}\left(X\right) = \mathbb{E}\left(X^2\right) - \mathbb{E}\left(X\right)^2.$$

Proof.

$$\begin{aligned}
\mathrm{Var}\left(X\right) &= \mathbb{E}\left(\left(X - \mathbb{E}\left(X\right)\right)^2\right) \\
&= \mathbb{E}\left(X^2\right) - \mathbb{E}\left(2X\mathbb{E}\left(X\right)\right) + \mathbb{E}\left(\mathbb{E}\left(X\right)^2\right) \\
&= \mathbb{E}\left(X^2\right) - 2\,\mathbb{E}\left(X\right)\mathbb{E}\left(X\right) + \mathbb{E}\left(X\right)^2 \\
&= \mathbb{E}\left(X^2\right) - \mathbb{E}\left(X\right)^2.
\end{aligned}$$

$\square$

Example (1 of 2)

What is the variance in the number of female children in a family of 5 children?

Example (1 of 2)

What is the variance in the number of female children in a family of 5 children?

If X represents the number of female children in a family of five children, then

$$\mathrm{Var}(X) = \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2 = 7.5 - (2.5)^2 = 1.25\,.$$

Example (2 of 2)

Find the variance of a Bernoulli random variable for which the probability of success is p.

Example (2 of 2)

Find the variance of a Bernoulli random variable for which the probability of success is p.

If X represents the outcome of a Bernoulli trial, then

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2 \\
&= \left[1^2(p) + 0^2(1-p)\right] - p^2 \\
&= p - p^2 \\
&= p(1-p).
\end{aligned}$$

Variance of a Sum (1 of 3)

Theorem

Let $X_1, X_2, \ldots, X_k$ be pairwise independent random variables, then

$$\mathrm{Var}\,(X_1 + X_2 + \cdots + X_k) = \mathrm{Var}\,(X_1) + \mathrm{Var}\,(X_2) + \cdots + \mathrm{Var}\,(X_k)\,.$$

Variance of a Sum (2 of 3)

Take the case when $k = 2$.

$$\begin{aligned}
\operatorname{Var}(X+Y) &= \mathbb{E}\left(((X+Y) - \mathbb{E}(X+Y))^2\right) \\
&= \mathbb{E}\left(((X - \mathbb{E}(X)) + (Y - \mathbb{E}(Y)))^2\right) \\
&= \mathbb{E}\left((X - \mathbb{E}(X))^2\right) + \mathbb{E}\left((Y - \mathbb{E}(Y))^2\right) + \\
&\quad 2\mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) \\
&= \operatorname{Var}(X) + \operatorname{Var}(Y) \\
&\quad + 2\mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))
\end{aligned}$$

Variance of a Sum (2 of 3)

Since we are assuming that random variables X and Y are independent, then

$$\begin{aligned}
\mathbb{E}\left((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\right) &= \mathbb{E}(X - \mathbb{E}(X))\,\mathbb{E}(Y - \mathbb{E}(Y)) \\
&= (\mathbb{E}(X) - \mathbb{E}(X))(\mathbb{E}(Y) - \mathbb{E}(Y)) \\
&= 0,
\end{aligned}$$

and thus

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

The result can be extended to any finite value of k by induction.

Example

Find the variance of a Binomial random variable with n trials when the probability of success on a single trial is p.

Example

Find the variance of a Binomial random variable with n trials when the probability of success on a single trial is p.

Each trial of a binomial experiment is a Bernoulli experiment with probability of success p. Since the n trials of a binomial experiment are independent

$$\mathrm{Var}(X) = np(1-p).$$

Variance of a Geometric RV (1 of 2)

- The probability mass function of a geometric RV is given by

$$\mathbb{P}(X = n) = f_X(n) = (1-p)^{n-1} p \text{ for } n \in \mathbb{N}.$$

- Recall the geometric series summation,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ for } |z| < 1.$$

- Differentiate the summation twice.

$$\sum_{n=1}^{\infty} n(n-1)\, z^{n-2} = \frac{2}{(1-z)^3}$$

Variance of a Geometric RV (2 of 2)

$$\sum_{n=1}^{\infty} n(n-1)\, z^{n-2} = \frac{2}{(1-z)^3}$$

Replace z by $1-p$.

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2\,(1-p)^{n-2} - \sum_{n=1}^{\infty} n\,(1-p)^{n-2} &= \frac{2}{(1-(1-p))^3} \\
\sum_{n=1}^{\infty} n^2\,(1-p)^{n-1}p - \sum_{n=1}^{\infty} n\,(1-p)^{n-1}p &= \frac{2(1-p)p}{(1-(1-p))^3} \\
\mathbb{E}\left(X^2\right) - \frac{1}{p} &= \frac{2(1-p)}{p^2} \\
\mathbb{E}\left(X^2\right) - \mathbb{E}\,(X)^2 &= \frac{2(1-p)}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2 \\
\mathrm{Var}\,(X) &= \frac{1-p}{p^2}
\end{aligned}$$

Variance of a Product (1 of 2)

Theorem

Let $X_1, X_2, \ldots, X_k$ be pairwise independent random variables, then

$$\begin{aligned} \mathrm{Var}\left(X_1 X_2 \cdots X_k\right) &= \mathbb{E}\left(X_1^2\right)\mathbb{E}\left(X_2^2\right)\cdots\mathbb{E}\left(X_k^2\right) \\ &\quad - \left(\mathbb{E}\left(X_1\right)\mathbb{E}\left(X_2\right)\cdots\mathbb{E}\left(X_k\right)\right)^2 . \end{aligned}$$

Variance of a Product (2 of 2)

Proof.

$$\begin{aligned}
\operatorname{Var}(X_1 X_2 \cdots X_k) &= \mathbb{E}\left((X_1 X_2 \cdots X_k)^2\right) - (\mathbb{E}(X_1 X_2 \cdots X_k))^2 \\
&= \mathbb{E}\left(X_1^2 X_2^2 \cdots X_k^2\right) - (\mathbb{E}(X_1)\,\mathbb{E}(X_2) \cdots \mathbb{E}(X_k))^2 \\
&= \mathbb{E}\left(X_1^2\right) \mathbb{E}\left(X_2^2\right) \cdots \mathbb{E}\left(X_k^2\right) \\
&\quad - (\mathbb{E}(X_1)\,\mathbb{E}(X_2) \cdots \mathbb{E}(X_k))^2
\end{aligned}$$

□

Homework

- Read Sections 2.6, 2.7
- Exercises: 16, 18–25

Credits

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