Forwards on Dividend-Paying Assets and Transaction Costs

MATH 472 *Financial Mathematics*

J Robert Buchanan

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Objectives

In this lesson we will learn:

- how to price forward contracts on assets which pay dividends in an arbitrage-free setting. Discrete and continuous dividends will be handled.
- how to incorporate transaction costs into the elimination of arbitrage.

Incorporating Dividends

Remarks:

- **Dividends** are periodic payments to the owners of a security paid out of corporate profits.
- Dividends are paid to the *shareholders*, not to the owners of prepaid forwards or forward contracts.
- The price of a prepaid forward or forward contract must be discounted for any dividends paid during the time interval $[0, T]$.
- The amount of discount should be the present value of the dividend(s).

Prepaid Forwards on Dividend-Paying Stocks

Assume:

- risk-free interest rate, r
- dividends $\{D_1, D_2, \dots, D_n\}$ are paid at times $\{t_1, t_2, \dots, t_n\}$ in the interval $[0, T]$

Then the price of a prepaid forward on an asset currently valued at S_0 becomes

$$F_{0,T}^{P} = S_0 - \sum_{i=1}^{n} D_i e^{-r\, t_i}.$$

Continuous Dividends

Assume the asset pays dividends at a continuously compounded annual rate of δ.

If S_0 of the asset is owned at $t = 0$ and the dividends are reinvested in the index then at $t = T$, the amount of asset owned will be $S_0 e^{\delta T}$.

Thus if the asset pays dividends continuously at rate δ, then

$$F^P_{0,T} = S_0 e^{-\delta T}.$$

Forward Contracts on Dividend-Paying Stocks

Assume:

- risk-free interest rate, r
- dividends $\{D_1, D_2, \ldots, D_n\}$ are paid at times $\{t_1, t_2, \ldots, t_n\}$ in the interval $[0, T]$

Then the price of a forward contract on a stock currently valued at S_0 becomes

$$F_{0,T} = S_0 e^{r\,T} - \sum_{i=1}^{n} D_i e^{r(T-t_i)}.$$

Forward Contracts on Dividend-Paying Stocks

Assume:

- risk-free interest rate, r
- dividends $\{D_1, D_2, \ldots, D_n\}$ are paid at times $\{t_1, t_2, \ldots, t_n\}$ in the interval $[0, T]$

Then the price of a forward contract on a stock currently valued at S_0 becomes

$$F_{0,T} = S_0 e^{r\,T} - \sum_{i=1}^{n} D_i e^{r(T-t_i)}.$$

If the stock pays dividends continuously at rate δ, then

$$F_{0,T} = S_0 e^{(r-\delta)T}.$$

Example (1 of 4)

Suppose the risk-free interest rate is 5.05%. A share of stock whose current value is \$110 per share will pay a dividend in six months of \$5 and another in twelve months of \$8. Find the prices of a one-year forward contract and one-year prepaid forward on the stock assuming that transfer of ownership will take place immediately after the second dividend is paid.

Example (2 of 4)

The value of the prepaid forward is

$$F^P_{0,T} = 110 - 5e^{-0.0505(6/12)} - 8e^{-0.0505(12/12)} \approx 97.5186.$$

The value of a forward contract on the dividend paying stock is

$$F_{0,T} = 97.5186e^{0.0505(12/12)} \approx 102.57.$$

Example (3 of 4)

An investment valued at \$125 pays dividends continuously at the annual rate of 2.75%. The risk-free interest rate is 3.5%. Find the prices of a four-month prepaid forward and a four-month forward contract on the investment.

Example (4 of 4)

The price of a four-month prepaid forward on the investment is

$$F^P_{0,T} = 125e^{-0.0275(4/12)} \approx 123.859.$$

The value of a four-month forward contract on the investment is

$$F_{0,T} = 125e^{(0.035-0.0275)(4/12)} \approx 125.313.$$

Terminology

Market maker: an agent who arranges trades between buyers and sellers.

Bid price: amount a buyer is willing to spend for an item.

Ask price: amount a seller is willing to accept for an item. The ask price is also known as the **offer price**.

Bid/Ask spread: difference between the bid and ask prices for the same item.

Example

- Suppose the lowest ask price of a share of stock is $50.10 and the highest bid price for the stock is $50.00.
- The bid/ask spread is therefore $0.10 per share.
- A stock buyer who issues a buy order for 1000 shares will pay $50,100.
- The seller will receive $50,000 and the market maker will earn $100 on the trade (plus any other fees or commissions charged).

Incorporating Transaction Costs

- S^a: the time $t = 0$ ask price at which the security can be bought.
- S^b: the time $t = 0$ bid price at which the security can be sold. In general $S^b < S^a$.
- r^b: the continuously compounded interest rate at which money may be borrowed.
- r^l: the continuously compounded interest rate at which money may be lent. In general $r^l < r^b$.
- k: the cost per transaction for executing a purchase or sale.

Pricing a Forward Contract

Theorem

The arbitrage-free forward contract price must satisfy the inequality

$$F^{-} \equiv (S^{b} - 2k)e^{r^{l}T} \leq F \leq (S^{a} + 2k)e^{r^{b}T} \equiv F^{+}.$$

Proof (1 of 2)

Define $F^+ = (S^a + 2k)e^{r^b T}$.

Assumption: $F > F^+$

1. At time $t = 0$ an investor may borrow amount $S^a + 2k$ to purchase the security and sell the forward contract. The net cash flow at time $t = 0$ is zero.
2. At time $t = T$ the loan must be repaid in the amount of $(S^a + 2k)e^{r^b T}$ and the investor receives F for the forward. The total cash flow for times $t = 0$ and $t = T$ is therefore

$$F - (S^a + 2k)e^{r^b T} = F - F^+ > 0.$$

Proof (2 of 2)

Now define $F^- = (S^b - 2k)e^{r^l T}$.

Assumption: $F < F^-$

1. At time $t = 0$ an investor can purchase the forward contract and sell short the security for S^b. A transaction cost of k is paid at time $t = 0$ for the forward contract and another transaction cost of k is incurred during the short sale. The net proceeds from the sale are $S^b - 2k$. This amount is lent out at interest rate r^l until time $t = T$.
2. At time $t = T$ the investor's cash balance is $(S^b - 2k)e^{r^l T}$. The investor pays F for the forward contract and closes out the short position in the security. Thus the total cash flow at times $t = 0$ and $t = T$ is

$$(S^b - 2k)e^{r^l T} - F = F^- - F > 0.$$

Example (1 of 2)

Suppose the asking price for a certain stock is \$55 per share, the bid price is \$54.50 per share, the fee for buying or selling a share or a forward contract is \$1.50 per transaction, the continuously compounded lending rate is 2.5% per year, and the continuously compounded borrowing rate is 5.5% per year. Find the interval of no-arbitrage prices for a three-month forward contract on the stock.

Example (2 of 2)

$$
\begin{aligned}
(S^b - 2k)e^{r^l T} &\leq F \leq (S^a + 2k)e^{r^b T} \\
(54.50 - 2(1.50))e^{0.025(3/12)} &\leq F \leq (55 + 2(1.50))e^{0.055(3/12)} \\
51.7223 &\leq F \leq 58.8030
\end{aligned}
$$

Homework

- Read Sections 6.3, 6.4
- Exercises: 5–11

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

author: J. Robert Buchanan

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