Introduction to Forward Contracts

MATH 472 *Financial Mathematics*

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Objectives

In this lesson we will learn:

- the definitions of financial instruments known as forward contracts,
- study how forward contracts are traded, and
- how to price forward contracts in an arbitrage-free setting.

Steps Involved in an Asset Purchase

To purchase an asset outright requires:

1. setting the price to be paid,
2. transferring the cash from buyer to seller,
3. transferring the asset from the seller to buyer.

Remarks:

- If these three steps occur essentially at the same time this situation is called **outright purchase**.
- When step 1 occurs at a given time and steps 2 and 3 occur at a later time, a **forward contract** has been formed.

Forwards (1 of 2)

Definition

A **forward contract** (or **forward**) is an agreement between two parties to buy or sell a specified quantity of a commodity at a specified price on a specified date in the future.

Remark: a forward obligates the parties to buy/sell.

Forwards (2 of 2)

A **forward contract** sets today the terms at which something will be bought or sold at some time in the future. It specifies the following aspects of the transaction:

- the type and quantity of the asset or commodity the seller must deliver (the **underlying asset**),
- the delivery logistics (date, time, place (the **expiration date**)),
- the price the buyer will pay at the time of delivery,
- obligations of the buyer and seller.

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- obligations of the buyer and seller.

Remarks:

- A forward contract requires no initial payment or premium.
- A forward contract binds both buyer and seller even if the future value of the asset or commodity is less than the market price.

Reading Price Quotations

FIGURE 2.1

Index futures price listings.

Index Futures

	OPEN	HIGH	LOW	SETTLE	CHG	LIFETIME HIGH	LIFETIME LOW	OPEN INT
DJ Industrial Average (CBT)-$10 x index								
Sept	9953	10085	9950	10062	113	10557	9835	40,620
Dec	10072	10080	10005	10059	116	10575	8440	349
Est vol 7,149; vol Mon 8,914; open int 40,969, +132.								
Idx prl: Hi 10103.13; Lo 9963.54; Close 10085.14, +123.22.								
Mini DJ Industrial Average (CBT)-$5 x index								
Sept	9956	10087	9949	10062	113	10629	9840	48,695
Vol Tue 105,733; open int 48,789, +1,159.								
DJ-AIG Commodity Index (CBT)-$100 x index								
Aug	...	...	...	455.9	...	485.0	449.4	2,886
Est vol 0; vol Mon 0; open int 2,886, unch.								
Idx prl: Hi 144.743; Lo 144.024; Close 144.341, −.019.								
S&P 500 Index (CME)-$250 x index								
Sept	108310	109600	108300	109250	960	116080	78100	575,947
Dec	109350	109550	109300	109290	970	116010	78100	12,717
Est vol 39,263; vol Mon 37,876; open int 589,488, −1,470.								
Idx prl: Hi 1096.65; Lo 1084.07; Close 1094.83, +10.76.								
Mini S&P 500 (CME)-$50 x index								
Sept	108300	109600	108275	109250	950	114850	107500	596,759
Vol Tue 729,906; open int 642,116, +334.								
S&P Midcap 400 (CME)-$500 x index								
Sept	567.00	575.00	567.00	573.40	7.15	616.50	508.70	13,640
Est vol 496; vol Mon 700; open int 13,640, +49.								
Idx prl: Hi 574.85; Lo 566.31; Close 574.02, +7.63.								
Nasdaq 100 (CME)-$100 x index								
Sept	137300	139800	137250	139050	1750	156500	136000	70,024
Est vol 12,432; vol Mon 12,422; open int 72,319, −20.								
Idx prl: Hi 1395.95; Lo 1371.26; Close 1391.50, +23.10.								

Source: Wall Street Journal, July 28, 2004, p. C-16.

Robert L. McDonald, *Derivatives Markets*, 2nd edition, Pearson Education, Inc., Boston, MA, USA (2006).

Example

A restaurant owner needs thirty cases of champagne for a New Year's Eve party. Knowing that a large quantity of champagne may be difficult to obtain (at a reasonable price) at the end of December, the restaurant owner may enter into a forward agreement with a supplier. The terms of the forward would indicate the quantity, price, and terms of delivery for the champagne.

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Question: what if the restaurant owner, after entering into the forward agreement cancels the party and thus does not need the champagne?

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Question: what if the restaurant owner, after entering into the forward agreement cancels the party and thus does not need the champagne?
Answer: the owner may sell the forward contract to someone else in order to recoup his cost.

Variations on the Purchase Process

outright purchase: three events occur simultaneously

fully leveraged purchase: set price and receive ownership at $t = 0$, pay for purchase at $t = T > 0$.

prepaid forward: set price and pay for purchase at $t = 0$, receive ownership at $t = T > 0$.

forward contract: set price at $t = 0$, pay for purchase and receive ownership at $t = T > 0$.

Long and Short Positions

long position: the owner of an item is in the "long position"

short position: the seller of an item is in the "short position"

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Remark: it is common for agents in the market to borrow assets to sell now with plans to re-purchase later (to return what was borrowed) when the price is lower (they hope).

Example of Short Selling

- Agent A believes the price of a stock will decrease during the next 30 days.
- Agent A borrows from Agent B a share of stock and sells it for S_0 with the agreement that the stock must be returned to Agent B by $t = 30$.
- If $S_t < S(0)$ for some $0 < t \leq 30$, Agent A purchases the stock for S_t and returns it to Agent B.
- Agent A keeps a net profit of $S_0 - S_t > 0$.

Payoff of Forward Contract

Every forward contract has a buyer and a seller.

- The buyer is in the **long** position and benefits if the price of the underlying increases. The **payoff** of a long forward is

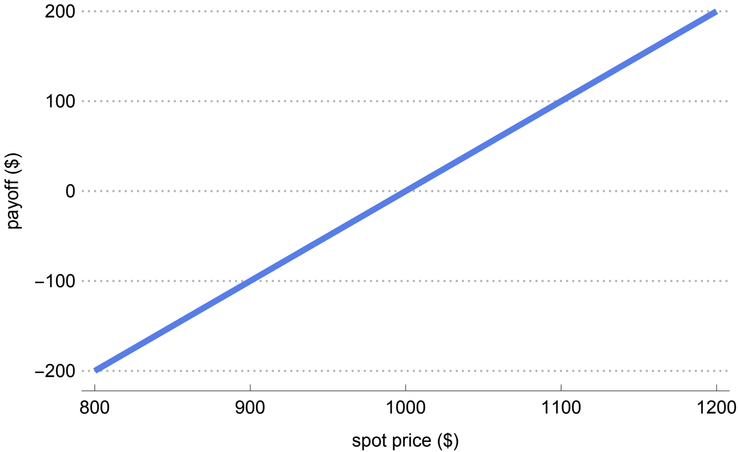
$$\text{payoff} = \text{spot price at expiry} - \text{forward price}.$$

- The seller is in the **short** position and benefits if the price of the underlying decreases. The **payoff** of a short forward is

$$\text{payoff} = \text{forward price} - \text{spot price at expiry}.$$

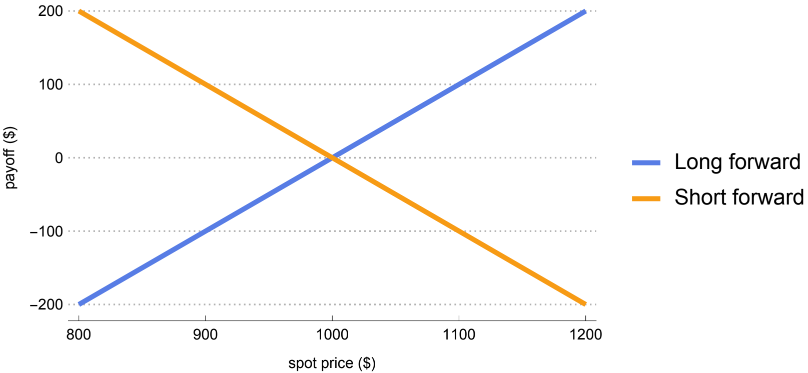
Illustration

Suppose an investor has a long forward on the S&R 500 index for $1000. The payoff has a graph which resembles that below.



Compare Long and Short Positions

Imagine a forward contract on the S&R 500 index for $1000.



Forward *vs*. Outright Purchase (1 of 3)

Suppose we compare the outright purchase of the S&R 500 index for $970 and a long forward expiring in 6 months for $1000.

- With outright purchase we pay $970 now and own the index for the next 6 months.
- With a long forward we are obligated to purchase the index in 6 months for $1000 (a great deal can happen in 6 months).

Forward *vs.* Outright Purchase (1 of 3)

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Suppose we sell the S&R 500 index after 6 months. What is our payoff?

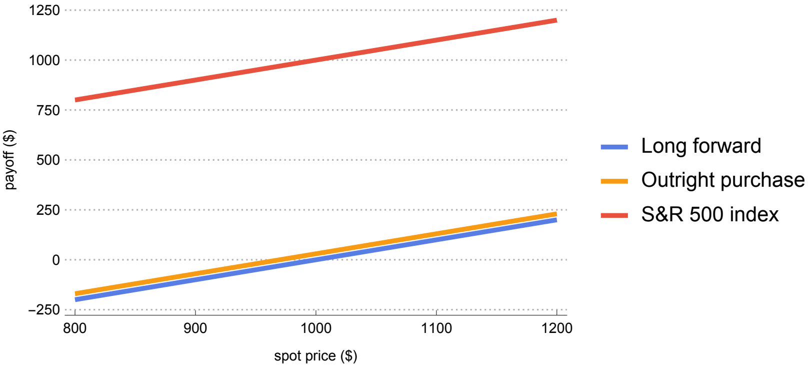
Forward *vs*. Outright Purchase (2 of 3)

Compare the cashflows assuming that the S&R 500 Index will be worth S_T in 6 months.

- **Outright purchase:** $S_T - 970$.
- **Long forward:** $S_T - 1000$.

Note: the \$970 for the outright purchase was paid at $t = 0$ while the \$1000 for the long forward was paid at $t = T > 0$.

Forward *vs.* Outright Purchase (3 of 3)



This comparison fails to consider the future value of the initial investment of $970 required for the outright purchase of the S&R 500 index.

Payoff *vs.* Profit

Payoff: describes the cash value of a position at a point in time.

Profit: describes the cash value of a position minus the future value of the initial investment in the position.

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Cash settlement: rather than a physical transfer of assets at expiry, forward contracts are sometimes settled for cash to avoid additional transaction costs.

Fair Comparison (1 of 2)

In order to fairly compare the outright purchase and the long forward we must invest the same amounts initially and account for interest earned or paid over 6 months.

Outright Purchase

- Purchase the index for $970.

Long forward

- Loan $970 for 6 months.
- Long forward contract for $1000.

Fair Comparison (2 of 2)

Outright Purchase

$$\begin{aligned} \text{payoff} &= S_T \\ \text{profit} &= S_T - 970e^{r/2} \end{aligned}$$

Long forward

$$\begin{aligned} \text{payoff} &= S_T - 1000 + 970e^{r/2} \\ \text{profit} &= S_T - 970e^{r/2} \end{aligned}$$

If $r \approx 0.0609$ the payoffs of the outright purchase and the long forward are the same.

Forward Contract and a Loan

- Investing \$970 at 6.09% for 6 months and entering a long forward for \$1000 in 6 months is the same as the outright purchase of the index for \$970.
- Borrowing \$970 at 6.09% for 6 months and purchasing the index for \$970 is the same as entering a 6-month long forward contract for \$1000.

Pricing Purchases

Theorem (Outright Purchase)

If an asset is worth S_0 at time $t = 0$ and payment and transfer of ownership will take place at time $t = 0$ then the amount paid should be S_0.

Theorem (Fully Leveraged Purchase)

If the continuously compounded interest rate is r, if an asset is worth S_0 at time $t = 0$, transfer of ownership will take place at time $t = 0$, and payment will be made at time $t = T > 0$, the amount of payment will be $S_0 e^{rT}$.

Pricing a Prepaid Forward Contract

Theorem

The price $F^P_{0,T}$ *of a prepaid forward contract on a non-dividend paying asset initially worth* S_0 *at time* $t = 0$ *for which ownership of the asset will be transferred to the buyer at time* $t = T > 0$ *is* $F^P_{0,T} = S_0$.

No Arbitrage Proof (1 of 2)

Assumption: $F^P_{0,T} < S_0$.

1. Purchase the forward and sell (short) the asset. Since $S_0 - F^P_{0,T} > 0$, there is a positive cash flow at $t = 0$.
2. At $t = T$, the buyer receives ownership of the asset and immediately closes (unwinds) their short position in the asset. The cash flow at $t = T$ is therefore zero.
3. Thus the total cash flows at $t = 0$ and $t = T$ is $S_0 - F^P_{0,T} > 0$.

There is no risk since the forward obligates the seller to deliver the asset to the buyer so that the buyer's short position in the asset can be closed out.

No Arbitrage Proof (2 of 2)

Assumption: $F^P_{0,T} > S_0$.

1. Purchase the asset at time $t = 0$ and sell a prepaid forward. Since $F^P_{0,T} - S_0 > 0$ there is a positive cash flow at $t = 0$.
2. At $t = T$, the buyer must transfer ownership of the asset to the party who purchased the forward. The cash flow at $t = T$ is therefore zero.
3. Thus the total cash flows at $t = 0$ and $t = T$ is $F^P_{0,T} - S_0 > 0$.

There is no risk in this situation since the buyer owns the asset at time $t = 0$ and thus will with certainty be able to transfer ownership at $t = T$.

Pricing a Forward Contract

Theorem

Suppose a share of a non-dividend paying asset is worth S_0 *at time* $t = 0$ *and that the continuously compounded risk-free interest rate is* r*, then the price of the forward contract is*

$$F_{0,T} = S_0 e^{rT}.$$

Proof (1 of 2)

Assumption: $F_{0,T} < S_0 e^{rT}$.

1. The buyer can purchase the forward and sell the asset at time $t = 0$.
2. The value of the asset is S_0 which is lent out at the risk-free rate compounded continuously. Thus the net cash flow at time $t = 0$ is $S_0 - S_0 = 0$. At $t = T$, when the borrower repays the loan, the buyer's cash balance is $S_0 e^{rT}$.
3. The buyer pays $F_{0,T}$ for the forward in order to receive the asset which is then used to close out the short position. The cash flow at $t = T$ is therefore $-F_{0,T}$. Thus the total cash flows at $t = 0$ and $t = T$ is $S_0 e^{rT} - F_{0,T} > 0$.

There is no risk in obtaining this positive profit since the forward obligates the seller to deliver the asset to the buyer so that the buyer's short position in the asset can be closed out.

Proof (2 of 2)

Assumption: $F_{0,T} > S_0 e^{rT}$.

1. The buyer can sell a forward contract and borrow S_0 to purchase the asset at time $t = 0$. Thus the net cash flow at time $t = 0$ is $S_0 - S_0 = 0$.
2. At $t = T$, the buyer must repay the loan of $S_0 e^{rT}$ and will sell the asset for $F_{0,T}$. The cash flow at $t = T$ is therefore $F_{0,T} - S_0 e^{rT} > 0$. Thus the total cash flows at $t = 0$ and $t = T$ is $F_{0,T} - S_0 e^{rT} > 0$.

There is no risk in this situation since the buyer owns the asset at time $t = 0$ and thus will with certainty be able to transfer ownership at $t = T$.

Profit

Definition

The **profit** on a forward contract is

$$\text{profit} = S_T - F_{0,T} = S_T - S_0 e^{r\,T}.$$

This is the net amount of money gained/lost when the stock is sold on the delivery date.

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Remark: we subtract the *future value* of S_0, **not** the *present value* of S_0. We will apply the same principle later when calculating the profit on any financial position taken by an investor.

Example

Suppose a share of stock is currently trading for \$25 and the risk-free interest rate is 4.65% per year. Find the price of a two-month forward contract.

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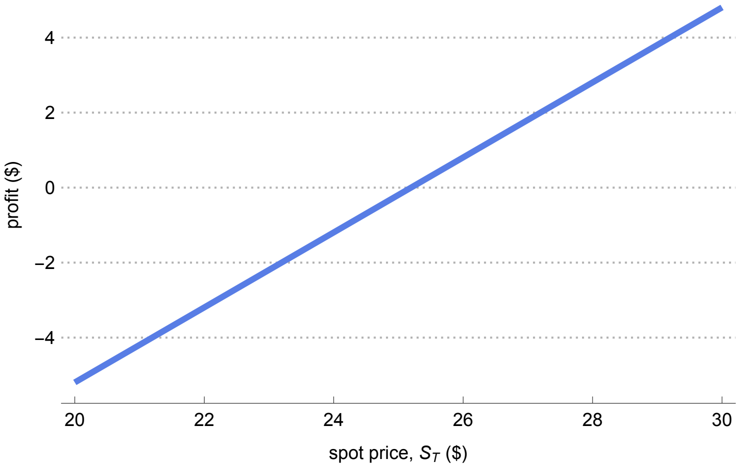
The price of a two-month forward contract is

$$F = 25e^{0.0465(2/12)} \approx 25.1945.$$

The profit is then $S(2/12) - 25.1945$.

Illustration



Homework

- Read Sections 6.1, 6.2
- Exercises: 2–4

Credits

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